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MISSILE IMPACT ERRORS DUE TO NAVIGATIONAL AND IN-FLIGHT EFFECTS OF ANOMALOUS GRAVITY

Oscar T. Schultz

U.S. NAVAL WEAPONS LABORATORY
DAHLGREN, VIRGINIA

NWL

NAVAL WEAPONS LABORATORY
Dahlgren, Virginia
22448

R. B. Meeks, Jr., Capt., USN
Commander

James E. Colvard
Technical Director

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by

Oscar T. Schultz
Warfare Analysis Department

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William R. Elsaesser has contributed to several portions of the analysis and has checked all of the analysis in detail. He programmed independently an earlier abbreviated version of the computations and programmed independently as a check several portions of the final computations.

FOREWORD

This report presents the analytical methods and results of a study of missile impact errors produced by anomalous gravity. The report has been reviewed by Sharon K. Gripshover for Raymond H. Hughey, Jr., Head, FBM Geoballistics Division.

Released by:

RALPH A. NIEMANN
Head, Warfare Analysis Department

ABSTRACT

This report presents an analysis of the impact errors of a ship-launched ballistic missile produced by an anomalous gravity field. All other error sources are assumed to be absent. The normal earth for which the impact errors would be zero is regarded as spherical and non-rotating. The standard atmosphere is accounted for. Both navigation and in-flight effects are considered, the former under the assumption that the ship remains stationary at the launch point.

The treatment is largely deterministic, with elementary statistical concepts being introduced to derive a description of the impact errors applicable to the case of an unknown anomalous field.

Some numerical statistical results are presented, in the form of a table for a particular trajectory and as graphs for trajectories having various ranges and times of flight.

Possible extensions of the analysis are discussed.

CONTENTS

I. INTRODUCTION

This report presents the analytical methods and results of an investigation to determine the extent of gravity data required for a ship-launched ballistic missile to achieve specified impact accuracy. The position and velocity of the ship at missile launching are assumed to be derived from an inertial navigation system and the trajectory of the missile is assumed to be controlled by an inertial guidance system. Since the operation of both of these inertial systems depends on knowledge of the earth's gravity field, incomplete or erroneous knowledge of this field leads to both navigation and in-flight errors which produce the resultant impact errors.

To allow a concentration on gravity effects without the complications that would be introduced by a simultaneous consideration of other error sources, it is assumed that the ship's inertial navigation system and the missile's inertial guidance system are perfect in all respects except for the omission of information on the anomalous gravity field. More specifically, it is supposed that if the field were strictly normal, then this normal field could be taken into account completely and no impact errors would result. The impact errors considered are therefore those which result because the anomalous field is unknown or, if known, is deliberately not accounted for.

It is, of course, impossible to make definitive statements about the impact errors produced by an unknown anomalous field. The analysis therefore proceeds as though the anomalous field were known but is deliberately not taken into account in navigation and guidance. Formulas are then obtained which would permit the computation of the resulting impact error for a specific set of launch conditions and a specific set of parameters describing the anomalous field. By averaging these results over a certain ensemble of launch conditions a statistical description of the impact errors is obtained. This description is found to have such a form that it does not depend on a detailed description of the anomalous field but only on the partial description contained in the so-called "degree variances" of geoidal heights or, equivalently, on the "autocovariance function" of geoidal heights. It is this fact which allows the results to be given a plausible interpretation for the case where the anomalous field has not been taken into account because it is not completely known, since it is not unreasonable to assume that the degree variances or the autocovariance function may be approximately known even though a complete description of the anomalous field is unavailable.

The statistical results can also be interpreted as describing the impact errors that would result if a known part of the anomalous field (rather than merely the normal field) were taken into account in navigation and guidance and the remainder

of the anomalous field were neglected. Other interpretations of the results will be explained when the results are presented.

In addition to assuming that the navigation and guidance systems contain perfect components, it is supposed that the ship remains stationary for a considerable period (one or two hours) prior to launching the missile. This situation does not seem unrealistic and leads to a simplification of the analysis in which certain dynamic effects in the ship's inertial navigation system need not be considered. It may be possible to avoid this restriction by an extension of the methods to be described.

It is believed that the results obtained give rather reliable estimates of the minimum amount of gravity data which would be required to achieve specified impact errors. It would be expected that imperfect components in the navigation and guidance systems, and a non-zero speed of the ship, would lead to larger impact errors than those derived here, but that such a degradation could be alleviated by the employment of more extensive gravity data.

Other analyses of this problem have taken into account the speed of the ship and the use in the navigation system of imperfect components (particularly gyros, necessitating a consideration of navigation resets). However, these treatments require much more elaborate statistical techniques whose validity is difficult to evaluate. One of the objectives of the present investigation has been to employ deterministic mathematical methods to the greatest extent possible and then to introduce elementary statistical concepts only in the final stages.

II. THE ANOMALOUS FIELD

The anomalous gravity field which is the source of the errors considered in this report can be described in various ways. That which seems to be most satisfying intuitively, and is chosen here, is the geoidal height function $N(\theta,\lambda)$ of geodetic colatitude θ and longitude λ which gives the height of the geoid above the reference ellipsoid of revolution corresponding to the normal gravity field. The rotation of the earth introduces many complications into the computation of impact errors but, it is believed, does not contribute significantly to the impact errors produced by the anomalous field. To avoid these complications the rotation of the earth is neglected in this investigation. This is done by interpreting the given geoidal height function $N(\theta,\lambda)$ as defining, at geocentric colatitude θ and longitude λ, the height of a non-rotating geoid above a non-rotating reference sphere; and then replacing the normal gravity field corresponding to the rotating reference ellipsoid by an inverse square gravitational field. With these simplifications, it is supposed that the navigation and guidance systems account perfectly for the inverse square field but do not account for the anomalous field corresponding to the geoidal height function.

Positions are defined, as illustrated in Figure 1, in terms of spherical coordinates r, θ, λ or the corresponding rectangular coordinates

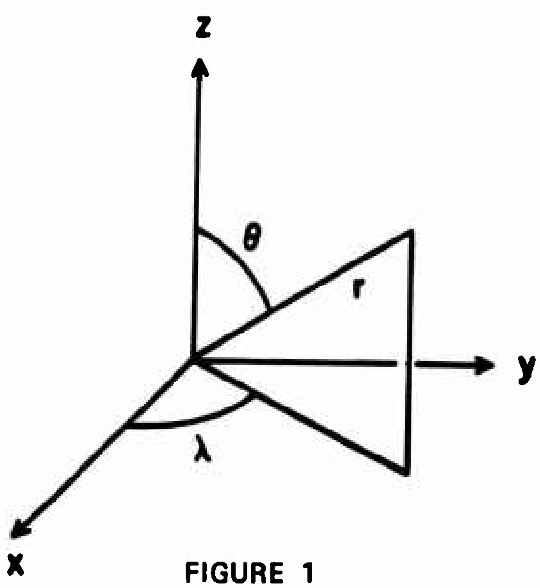

FIGURE 1

$$
\begin{aligned}
x &= r \sin\theta \cos\lambda \\
y &= r \sin\theta \sin\lambda \\
z &= r \cos\theta .
\end{aligned}
\tag{1}
$$

The origin is at the center of mass of the earth, the z-axis is directed toward the north pole, and the x-axis is in the Greenwich Meridian.

The gravitational potential of the earth will be denoted by $U(r,\theta,\lambda)$ or, in rectangular coordinates, by $u(x,y,z)$. The spherical coordinate form of the potential function can be expanded in a series of solid spherical harmonics

$$U(r,\theta,\lambda) = \frac{\mu}{r}\left[1 + \sum_{n=2}^{\infty}\left(\frac{R}{r}\right)^n \sum_{m=-n}^{+n} \alpha_n^m Y_n^m(\theta,\lambda)\right]$$

where

μ = earth's gravitational constant
R = a reference length taken as the radius of the reference sphere
α_n^m = complex coefficients defining the field, and
$Y_n^m(\theta,\lambda)$ are the complex surface spherical harmonics defined by

$$Y_n^m(\theta,\lambda) = (-1)^m \sqrt{\frac{2n+1}{4\pi} \frac{(n-m)!}{(n+m)!}} \, P_n^m(\cos\theta) e^{im\lambda} \tag{2}$$

in which $P_n^m(z)$ is the associated Legendre function

$$P_n^m(z) = \frac{1}{2^n n!}(1-z^2)^{\frac{m}{2}} \frac{d^{n+m}(z^2-1)^n}{dz^{n+m}} \quad . \tag{3}$$

The functions $Y_n^m(\theta,\lambda)$ satisfy the relations $Y_n^{-m}(\theta,\lambda) = (-1)^m \overline{Y_n^m(\theta,\lambda)}$ where the bar denotes the complex conjugate. From this and the fact that U is real valued, it follows that the complex coefficients must satisfy the conditions $\alpha_n^{-m} = (-1)^m \overline{\alpha_n^m}$. In the sum over n the terms for n = 1 have been omitted as a consequence of the origin's being at the earth's center of mass. The limits of the summation indices, and the indices themselves, are often omitted in what follows when they can be inferred from the context. It should be admitted that the infinite sum $\sum_{n=2}^{\infty}$ raises questions of convergence in some of the following operations. In practice, however, the potential is never known exactly and will be approximated by a finite sum for which questions of convergence do not arise. The numerical values of μ and R adopted here are

$$\mu = 3.986012 \times 10^{14} \text{ m}^3/\text{sec}^2$$

$$R = 6.371008608 \times 10^6 \text{ m.}$$

The nautical mile, used only in describing the range of a trajectory, is regarded as the distance on a sphere of radius R subtended by a central angle of one minute of arc.

The potential of the normal (inverse square) field is

$$v(x,y,z) = V(r,\theta,\lambda) = \frac{\mu}{r} .$$

The disturbing potential is

$$w(x,y,z) = u(x,y,z) - v(x,y,z)$$

or. in spherical coordinates,

$$\begin{aligned} W(r,\theta,\lambda) &= U(r,\theta,\lambda) - V(r,\theta,\lambda) \\ &= \frac{\mu}{r} \sum_{n=2}^{\infty} \left(\frac{R}{r}\right)^n \sum_{m=-n}^{+n} \alpha_n^m Y_n^m(\theta,\lambda) . \end{aligned} \tag{4}$$

The geoid is defined as the surface on which the actual potential $U(r,\theta,\lambda)$ has the same value that the normal potential $V(r,\theta,\lambda)$ has on the surface of the reference sphere. The equation of the geoid is therefore

$$\frac{\mu}{R} = \frac{\mu}{r}\left[1 + \sum_{n=2}^{\infty} \left(\frac{R}{r}\right)^n \sum_{m=-n}^{+n} \alpha_n^m Y_n^m(\theta,\lambda)\right] .$$

Let

$$r = R + N = R\left(1 + \frac{N}{R}\right)$$

where $N = N(\theta,\lambda)$ is the geoidal height. In terms of N the equation of the geoid is then

$$\frac{\mu}{R} = \frac{\mu}{R\left(1+\frac{N}{R}\right)}\left[1+\sum_{n=2}^{\infty}\left(1+\frac{N}{R}\right)^{-n}\sum_{m=-n}^{+n}\alpha_n^m Y_n^m(\theta,\lambda)\right]$$

or

$$\frac{N}{R} = \sum_{n=2}^{\infty}\left(1+\frac{N}{R}\right)^{-n}\sum_{m=-n}^{+n}\alpha_n^m Y_n^m(\theta,\lambda) .$$

Throughout the remainder of this report the coefficients α_n^m are regarded as small quantities of the first order whose products and powers can be neglected. With this approximation,

$$N = R\sum_{n=2}^{\infty}\sum_{m=-n}^{+n}\alpha_n^m Y_n^m(\theta,\lambda) . \tag{5}$$

If the coefficients α_n^m in the expansion of the potential are given, values of $N(\theta,\lambda)$ can be computed from this equation. If, on the other hand, values of the geoidal height function $N(\theta,\lambda)$ are given over the surface of the reference sphere, then the coefficients α_n^m can be computed from

$$\alpha_n^m = \frac{1}{R}\int_{\lambda=0}^{2\pi}\int_{\theta=0}^{\pi} N(\theta,\lambda)\overline{Y_n^m(\theta,\lambda)}\sin\theta\, d\theta\, d\lambda . \tag{6}$$

The geoidal height function is arbitrary except for the restrictions that the resulting values of $\alpha_0^0 = \alpha_1^{-1} = \alpha_1^0 = \alpha_1^1 = 0$. The condition $\alpha_0^0 = 0$ implies that the mean value of $N(\theta,\lambda)$ over the surface of the sphere is zero. The three conditions $\alpha_1^{-1} = \alpha_1^0 = \alpha_1^1 = 0$ imply that the centroid of the volume enclosed by the geoid is at the origin of the coordinate system.

III. EQUATIONS OF VARIATION

A nominal missile is defined as one which is launched from a known position on the reference sphere, flies in the normal gravitational field, and has zero impact error. The first situation considered is that in which a nominal missile is launched from the north pole at a target in the Greenwich Meridian. Its equations of motion are

$$\ddot{x}_0 = a_x(t) + \left.\frac{\partial v}{\partial x}\right|_0$$

$$\ddot{y}_0 = 0 + \left.\frac{\partial v}{\partial y}\right|_0 .$$

$$\ddot{z}_0 = a_z(t) + \left.\frac{\partial v}{\partial z}\right|_0 .$$

where $a_x(t)$, $a_y(t)$, $a_z(t)$ are the components of the non-gravitational acceleration produced by thrust and aerodynamic forces, with $a_y(t) = 0$ because the nominal trajectory lies in the zx-plane. The quantities

$$\left.\frac{\partial v}{\partial x}\right|_0 , \quad \left.\frac{\partial v}{\partial y}\right|_0 , \quad \left.\frac{\partial v}{\partial z}\right|_0 ,$$

are the components of the normal gravitational acceleration at the position (x_0, y_0, z_0) of the missile. For the trajectories considered in the present study, the thrust is terminated above the atmosphere so that $a_x(t)$ and $a_z(t)$ are also zero from the end of powered flight until reentry into the atmosphere.

Corresponding to this nominal missile we consider the trajectory of an actual missile which is launched from the "indicated" north pole (that is, from the position at which the ship's inertial navigation system, as affected by the anomalous gravitational field, indicates that the ship is at the north pole of the reference sphere) and whose flight is affected by the anomalous field. A simplified model of the missile's inertial guidance system is assumed. In this model, the missile is assumed to be controlled during powered flight in such a manner that its components of non-gravitational acceleration are the same functions of time as are the corresponding components for the nominal missile. The equations of motion for the actual missile during powered flight are therefore

$$\ddot{x} = a_x(t) + \frac{\partial u}{\partial x}$$

$$\ddot{y} = 0 + \frac{\partial u}{\partial y}$$

$$\ddot{z} = a_z(t) + \frac{\partial u}{\partial z} .$$

From the termination of powered flight until reentry these equations still apply with $a_x(t) = a_z(t) = 0$, as in the case of the nominal missile. From reentry until impact, $a_x(t)$ and $a_z(t)$ for the actual missile are not the same functions as for the nominal missile, and $a_y(t)$ is not equal to zero.

Considering the portion of the flight from launch to reentry, let $\xi = x - x_0$, $\eta = y - y_0$, $\zeta = z - z_0$ be the components of displacement of the actual missile from the nominal. The differential equations satisfied by ξ, η, ζ are then

$$\ddot{\xi} = \frac{\partial u}{\partial x} - \left.\frac{\partial v}{\partial x}\right|_0 = \frac{\partial v}{\partial x} - \left.\frac{\partial v}{\partial x}\right|_0 + \frac{\partial w}{\partial x}$$

$$\ddot{\eta} = \frac{\partial u}{\partial y} - \left.\frac{\partial v}{\partial y}\right|_0 = \frac{\partial v}{\partial y} - \left.\frac{\partial v}{\partial y}\right|_0 + \frac{\partial w}{\partial y}$$

$$\ddot{\zeta} = \frac{\partial u}{\partial z} - \left.\frac{\partial v}{\partial z}\right|_0 = \frac{\partial v}{\partial z} - \left.\frac{\partial v}{\partial z}\right|_0 + \frac{\partial w}{\partial z} .$$

If v and w are expanded as power series in ξ, η, ζ and terms involving powers and products of these quantities are neglected, the differential equations become

$$\ddot{\xi} = \left(\left.\frac{\partial^2 v}{\partial x^2}\right|_0 + \left.\frac{\partial^2 w}{\partial x^2}\right|_0\right)\xi + \left(\left.\frac{\partial^2 v}{\partial y \partial x}\right|_0 + \left.\frac{\partial^2 w}{\partial y \partial x}\right|_0\right)\eta + \left(\left.\frac{\partial^2 v}{\partial z \partial x}\right|_0 + \left.\frac{\partial^2 w}{\partial z \partial x}\right|_0\right)\zeta + \left.\frac{\partial w}{\partial x}\right|_0$$

$$\ddot{\eta} = \left(\left.\frac{\partial^2 v}{\partial x \partial y}\right|_0 + \left.\frac{\partial^2 w}{\partial x \partial y}\right|_0\right)\xi + \left(\left.\frac{\partial^2 v}{\partial y^2}\right|_0 + \left.\frac{\partial^2 w}{\partial y^2}\right|_0\right)\eta + \left(\left.\frac{\partial^2 v}{\partial z \partial y}\right|_0 + \left.\frac{\partial^2 w}{\partial z \partial y}\right|_0\right)\zeta + \left.\frac{\partial w}{\partial y}\right|_0$$

$$\ddot{\zeta} = \left(\left.\frac{\partial^2 v}{\partial x \partial z}\right|_0 + \left.\frac{\partial^2 w}{\partial x \partial z}\right|_0\right)\xi + \left(\left.\frac{\partial^2 v}{\partial y \partial z}\right|_0 + \left.\frac{\partial^2 w}{\partial y \partial z}\right|_0\right)\eta + \left(\left.\frac{\partial^2 v}{\partial z^2}\right|_0 + \left.\frac{\partial^2 w}{\partial z^2}\right|_0\right)\zeta + \left.\frac{\partial w}{\partial z}\right|_0 .$$

In the case of the normal gravitational field, the coefficients α_n^m are zero and the quantities ξ, η, ζ would be zero. This implies that the products $\alpha_n^m \xi$, etc., which would arise from the presence of the second partial derivatives of w in the right members are small quantities of order higher than the first and may therefore be neglected. With this simplification the equations become

$$\begin{aligned}
\ddot{\xi} &= \left.\frac{\partial^2 v}{\partial x^2}\right|_0 \xi + \left.\frac{\partial^2 v}{\partial y \partial x}\right|_0 \eta + \left.\frac{\partial^2 v}{\partial z \partial x}\right|_0 \zeta + \left.\frac{\partial w}{\partial x}\right|_0 \\
\ddot{\eta} &= \left.\frac{\partial^2 v}{\partial x \partial y}\right|_0 \xi + \left.\frac{\partial^2 v}{\partial y^2}\right|_0 \eta + \left.\frac{\partial^2 v}{\partial z \partial y}\right|_0 \zeta + \left.\frac{\partial w}{\partial y}\right|_0 \qquad (7) \\
\ddot{\zeta} &= \left.\frac{\partial^2 v}{\partial x \partial z}\right|_0 \xi + \left.\frac{\partial^2 v}{\partial y \partial z}\right|_0 \eta + \left.\frac{\partial^2 v}{\partial z^2}\right|_0 \zeta + \left.\frac{\partial w}{\partial z}\right|_0 \ .
\end{aligned}$$

As was stated earlier, the non-gravitational acceleration of the actual missile during reentry is not the same as that of the nominal missile and so, for this portion of the flight, the differential equations for ξ, η, ζ must be altered. This change requires a more detailed consideration of the trajectory. For the terminal portion of the flight of the actual missile it is assumed that the aerodynamic force consists only of a drag force D which acts in a direction opposite to the velocity. The equations of motion are then found to be

$$\begin{aligned}
\ddot{x} &= -\frac{D}{m}\frac{\dot{x}}{v} + \frac{\partial u}{\partial x} \\
\ddot{y} &= -\frac{D}{m}\frac{\dot{y}}{v} + \frac{\partial u}{\partial y} \\
\ddot{z} &= -\frac{D}{m}\frac{\dot{z}}{v} + \frac{\partial u}{\partial z} \ .
\end{aligned}$$

(Note that in these equations v represents the speed of the missile, not the normal gravitational potential. In some of the following equations v occurs with both meanings, but the proper interpretation will be clear from the context.) The corresponding equations for the nominal missile are

$$\ddot{x}_0 = -\frac{D_0}{m}\frac{\dot{x}_0}{v_0} + \left.\frac{\partial v}{\partial x}\right|_0$$

$$\ddot{y}_0 = -\frac{D_0}{m}\frac{\dot{y}_0}{v_0} + \left.\frac{\partial v}{\partial y}\right|_0$$

$$\ddot{z}_0 = -\frac{D_0}{m}\frac{\dot{z}_0}{v_0} + \left.\frac{\partial v}{\partial z}\right|_0 .$$

If ξ, η, ζ are defined as before, they satisfy the differential equations

$$\ddot{\xi} = -\frac{1}{m}\left(\frac{D\dot{x}}{v} - \frac{D_0\dot{x}_0}{v_0}\right) + \frac{\partial u}{\partial x} - \left.\frac{\partial v}{\partial x}\right|_0$$

$$\ddot{\eta} = -\frac{1}{m}\left(\frac{D\dot{y}}{v} - \frac{D_0\dot{y}_0}{v_0}\right) + \frac{\partial u}{\partial y} - \left.\frac{\partial v}{\partial y}\right|_0$$

$$\ddot{\zeta} = -\frac{1}{m}\left(\frac{D\dot{z}}{v} - \frac{D_0\dot{z}_0}{v_0}\right) + \frac{\partial u}{\partial z} - \left.\frac{\partial v}{\partial z}\right|_0 .$$

The differences $(\partial u/\partial x) - (\partial v/\partial x|_0)$, etc., are expanded as before to obtain the equations

$$\ddot{\xi} = -\frac{1}{m}\left(\frac{D\dot{x}}{v} - \frac{D_0\dot{x}_0}{v_0}\right) + \left.\frac{\partial^2 v}{\partial x^2}\right|_0 \xi + \left.\frac{\partial^2 v}{\partial y \partial x}\right|_0 \eta + \left.\frac{\partial^2 v}{\partial z \partial x}\right|_0 \zeta + \left.\frac{\partial w}{\partial x}\right|_0$$

$$\ddot{\eta} = -\frac{1}{m}\left(\frac{D\dot{y}}{v} - \frac{D_0\dot{y}_0}{v_0}\right) + \left.\frac{\partial^2 v}{\partial x \partial y}\right|_0 \xi + \left.\frac{\partial^2 v}{\partial y^2}\right|_0 \eta + \left.\frac{\partial^2 v}{\partial z \partial y}\right|_0 \zeta + \left.\frac{\partial w}{\partial y}\right|_0$$

$$\ddot{\zeta} = -\frac{1}{m}\left(\frac{D\dot{z}}{v} - \frac{D_0\dot{z}_0}{v_0}\right) + \left.\frac{\partial^2 v}{\partial x \partial z}\right|_0 \xi + \left.\frac{\partial^2 v}{\partial y \partial z}\right|_0 \eta + \left.\frac{\partial^2 v}{\partial z^2}\right|_0 \zeta + \left.\frac{\partial w}{\partial z}\right|_0 .$$

Now $(D\dot{x}/v) - (D_0\dot{x}_0/v_0) = \delta(D\dot{x}/v)$, etc., where the expression on the right denotes the increment of $D\dot{x}/v$ in passing from a point on the nominal trajectory to the corresponding (same time) point on the actual trajectory. Hence

$$\ddot{\xi} = -\frac{1}{m}\,\delta\left(\frac{D\dot{x}}{v}\right) + \left.\frac{\partial^2 v}{\partial x^2}\right|_0 \xi + \left.\frac{\partial^2 v}{\partial y \partial x}\right|_0 \eta + \left.\frac{\partial^2 v}{\partial z \partial x}\right|_0 \zeta + \left.\frac{\partial w}{\partial x}\right|_0$$

$$\ddot{\eta} = -\frac{1}{m}\,\delta\left(\frac{D\dot{y}}{v}\right) + \left.\frac{\partial^2 v}{\partial x \partial y}\right|_0 \xi + \left.\frac{\partial^2 v}{\partial y^2}\right|_0 \eta + \left.\frac{\partial^2 v}{\partial z \partial y}\right|_0 \zeta + \left.\frac{\partial w}{\partial y}\right|_0$$

$$\ddot{\zeta} = -\frac{1}{m}\,\delta\left(\frac{D\dot{z}}{v}\right) + \left.\frac{\partial^2 v}{\partial x \partial z}\right|_0 \xi + \left.\frac{\partial^2 v}{\partial y \partial z}\right|_0 \eta + \left.\frac{\partial^2 v}{\partial z^2}\right|_0 \zeta + \left.\frac{\partial w}{\partial z}\right|_0 .$$

Furthermore

$$\delta\left(\frac{D\dot{x}}{v}\right) = \delta\left(\frac{D}{v}\dot{x}\right) = \frac{D}{v}\,\delta\dot{x} + \dot{x}\,\delta\left(\frac{D}{v}\right)$$

$$= \frac{D_0}{v_0}\,\dot{\xi} + \dot{x}_0\,\delta\left(\frac{D}{v}\right)$$

where small quantities of higher order have been neglected. With similar approximations in the other two equations, the differential equations become

$$\ddot{\xi} = \left.\frac{\partial^2 v}{\partial x^2}\right|_0 \xi + \left.\frac{\partial^2 v}{\partial y \partial x}\right|_0 \eta + \left.\frac{\partial^2 v}{\partial z \partial x}\right|_0 \zeta - \frac{D_0}{m v_0}\,\dot{\xi} - \frac{\dot{x}_0}{m}\,\delta\left(\frac{D}{v}\right) + \left.\frac{\partial w}{\partial x}\right|_0$$

$$\ddot{\eta} = \left.\frac{\partial^2 v}{\partial x \partial y}\right|_0 \xi + \left.\frac{\partial^2 v}{\partial y^2}\right|_0 \eta \quad \left.\frac{\partial^2 v}{\partial z \partial y}\right|_0 \zeta - \frac{D_0}{m v_0}\,\dot{\eta} - \frac{\dot{y}_0}{m}\,\delta\left(\frac{D}{v}\right) + \left.\frac{\partial w}{\partial y}\right|_0$$

$$\ddot{\zeta} = \left.\frac{\partial^2 v}{\partial x \partial z}\right|_0 \xi + \left.\frac{\partial^2 v}{\partial y \partial z}\right|_0 \eta + \left.\frac{\partial^2 v}{\partial z^2}\right|_0 \zeta - \frac{D_0}{m v_0}\,\dot{\zeta} - \frac{\dot{z}_0}{m}\,\delta\left(\frac{D}{v}\right) + \left.\frac{\partial w}{\partial z}\right|_0 .$$

The drag force D is assumed to be of the form

$$D = \frac{1}{2}\rho v^2 S C_D$$

where ρ is the atmospheric density, C_D is the drag coefficient of the missile, and S is its cross section area. Hence, $D/v = (S/2)\rho v C_D$ and

$$\delta\left(\frac{D}{v}\right) = \frac{S}{2}\left(vC_D\,\delta\rho + \rho C_D\,\delta v + \rho v\delta C_D\right) .$$

Now $C_D = C_D(M)$ is a function of the Mach number $M = v/a$ in which a is the speed of sound. Hence

$$\delta C_D = C_D'(M)\delta M = \frac{1}{a}C_D'(\delta v - M\delta a)$$

where $C_D' = C_D'(M)$ is the derivative of C_D with respect to its argument M. Then $\rho v\delta C_D = \rho M C_D'(\delta v - M\delta a)$ and

$$\delta\left(\frac{D}{v}\right) = \frac{S}{2}\left[vC_D\,\delta\rho + \rho(C_D + MC_D')\delta v - \rho M^2 C_D'\,\delta a\right] .$$

Also $\rho = \rho(h)$ and $a = a(h)$ are functions of the height h so that $\delta\rho = \rho'(h)\delta h$ and $\delta a = a'(h)\delta h$, where the primes here denote derivatives with respect to h. With these substitutions

$$\delta\left(\frac{D}{v}\right) = \frac{S}{2}\left[(vC_D\rho' - \rho M^2 C_D' a')\delta h + \rho(C_D + MC_D')\delta v\right] ,$$

which can be abbreviated as $\delta\left(\frac{D}{v}\right) = \frac{mP}{v}\delta h + \frac{mQ}{v}\delta v$ in which

$$P = \frac{S}{2m}(vC_D\rho' - \rho M^2 C_D' a')v$$

$$Q = \frac{S}{2m}(C_D + MC_D')\rho v . \tag{8}$$

The height $h = r - R$ where r is the distance of the missile from the center of the earth and R is the radius of the reference sphere. Hence $\delta h = \delta r$ and, since $r^2 = x^2 + y^2 + z^2$,

$$\delta h = \frac{x}{r}\delta x + \frac{y}{r}\delta y + \frac{z}{r}\delta z .$$

Similarly, since $v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$,

$$\delta v = \frac{\dot{x}}{v}\delta\dot{x} + \frac{\dot{y}}{v}\delta\dot{y} + \frac{\dot{z}}{v}\delta\dot{z} .$$

Hence

$$\begin{aligned}\frac{1}{m}\delta\left(\frac{D}{v}\right) &= \frac{P}{v}\left(\frac{x}{r}\delta x + \frac{y}{r}\delta y + \frac{z}{r}\delta z\right) + \frac{Q}{v}\left(\frac{\dot{x}}{v}\delta\dot{x} + \frac{\dot{y}}{v}\delta\dot{y} + \frac{\dot{z}}{v}\delta\dot{z}\right)\\ &= \frac{P}{rv}(x\xi + y\eta + z\zeta) + \frac{Q}{v^2}(\dot{x}\dot{\xi} + \dot{y}\dot{\eta} + \dot{z}\dot{\zeta}) .\end{aligned}$$

Since one of the small quantities $\xi,\eta,\zeta,\dot{\xi},\dot{\eta},\dot{\zeta}$ appears in each term of this expression, the quantities $r,x,y,z,v,\dot{x},\dot{y},\dot{z},P,Q$, can be evaluated on the nominal trajectory for which $y = \dot{y} = 0$, and therefore

$$\frac{1}{m}\delta\left(\frac{D}{v}\right) = \frac{P}{rv}(x\xi + z\zeta) + \frac{Q}{v^2}(\dot{x}\dot{\xi} + \dot{z}\dot{\zeta}).$$

The differences between the position and velocity components of the actual and nominal missiles are now accounted for by the quantities $\xi,\eta,\zeta,\dot{\xi},\dot{\eta},\dot{\zeta}$; and the subscript zero can be omitted everywhere with the understanding that all quantities are to be evaluated on the nominal trajectory. The equations for ξ,η,ζ then become

$$\ddot{\xi} = \left(\frac{\partial^2 v}{\partial x^2} - P\frac{x}{r}\frac{\dot{x}}{v}\right)\xi + \frac{\partial^2 v}{\partial y \partial x}\eta + \left(\frac{\partial^2 v}{\partial z \partial x} - P\frac{z}{r}\frac{\dot{x}}{v}\right)\zeta$$
$$- \left(\frac{D}{mv} + Q\frac{\dot{x}}{v}\frac{\dot{x}}{v}\right)\dot{\xi} - Q\frac{\dot{x}}{v}\frac{\dot{z}}{v}\dot{\zeta} + \frac{\partial w}{\partial x}$$

$$\ddot{\eta} = \frac{\partial^2 v}{\partial x \partial y}\xi + \frac{\partial^2 v}{\partial y^2}\eta + \frac{\partial^2 v}{\partial z \partial y}\zeta - \frac{D}{mv}\dot{\eta} + \frac{\partial w}{\partial y}$$

$$\ddot{\zeta} = \left(\frac{\partial^2 v}{\partial x \partial z} - P\frac{x}{r}\frac{\dot{z}}{v}\right)\xi + \frac{\partial^2 v}{\partial y \partial z}\eta + \left(\frac{\partial^2 v}{\partial z^2} - P\frac{z}{r}\frac{\dot{z}}{v}\right)\zeta$$
$$- Q\frac{\dot{z}}{v}\frac{\dot{x}}{v}\dot{\xi} - \left(\frac{D}{mv} + Q\frac{\dot{z}}{v}\frac{\dot{z}}{v}\right)\dot{\zeta} + \frac{\partial w}{\partial z}.$$

Now let $x/r = \sin\theta$ and $z/r = \cos\theta$, where θ is the angle from the z-axis to the radius vector of the nominal missile; and let $\dot{x}/v = \sin\varphi$ and $\dot{z}/v = \cos\varphi$, where φ is the angle between the z-axis and the velocity vector of the nominal missile. Considering the normal potential $v = \mu/\sqrt{x^2 + y^2 + z^2}$, the partial derivatives of v appearing in the differential equations are found to be

$$\frac{\partial^2 v}{\partial x^2} = -\frac{\mu}{r^3}(1 - 3\sin^2\theta)$$

$$\frac{\partial^2 v}{\partial y^2} = -\frac{\mu}{r^3}$$

$$\frac{\partial^2 v}{\partial z^2} = -\frac{\mu}{r^3}(1 - 3\cos^2\theta)$$

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x} = 0$$

$$\frac{\partial^2 v}{\partial y \partial z} = \frac{\partial^2 v}{\partial z \partial y} = 0$$

$$\frac{\partial^2 v}{\partial z \partial x} = \frac{\partial^2 v}{\partial x \partial z} = \frac{3\mu}{r^3}\sin\theta\cos\theta.$$

The differential equations for ξ, η, ζ can therefore be written in the form

$$\ddot{\xi} = -\left[\frac{\mu}{r^3}(1 - 3\sin^2\theta) + P\sin\theta\sin\varphi\right]\xi + \left[\frac{3\mu}{r^3}\sin\theta\cos\theta - P\cos\theta\sin\varphi\right]\zeta$$
$$-\left[\frac{D}{mv} + Q\sin^2\varphi\right]\dot{\xi} - Q\sin\varphi\cos\varphi\,\dot{\zeta} + \frac{\partial w}{\partial x}$$

$$\ddot{\eta} = -\frac{\mu}{r^3}\eta - \frac{D}{mv}\dot{\eta} + \frac{\partial w}{\partial y}$$

$$\ddot{\zeta} = +\left[\frac{3\mu}{r^3}\sin\theta\cos\theta - P\sin\theta\cos\varphi\right]\xi - \left[\frac{\mu}{r^3}(1 - 3\cos^2\theta) + P\cos\theta\cos\varphi\right]\zeta$$
$$-Q\sin\varphi\cos\varphi\,\dot{\xi} - \left[\frac{D}{mv} + Q\cos^2\varphi\right]\dot{\zeta} + \frac{\partial w}{\partial z}$$

It is convenient to write these differential equations in matrix form by putting $\xi = x_1$, $\eta = x_2$, $\zeta = x_3$, $\dot{\xi} = x_4$, $\dot{\eta} = x_5$, $\dot{\zeta} = x_6$. This permits them to be written as the single differential equation

$$\frac{dx}{dt} = F(t)x + G(t) \tag{9}$$

where x and G(t) are the columns

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \qquad G(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial z} \end{bmatrix}$$

and F(t) is the 6 X 6 matrix

$$
F(t) = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
-\left[\frac{\mu}{r^3}(1 - 3\sin^2\theta) + P\sin\theta\sin\varphi\right] & 0 & \left[\frac{3\mu}{r^3}\sin\theta\cos\theta - P\cos\theta\sin\varphi\right] & -\left[\frac{D}{mv} + Q\sin^2\varphi\right] & 0 & -Q\sin\varphi\cos\varphi \\
0 & -\frac{\mu}{r^3} & 0 & 0 & -\frac{D}{mv} & 0 \\
\left[\frac{3\mu}{r^3}\sin\theta\cos\theta - P\sin\theta\cos\varphi\right] & 0 & -\left[\frac{\mu}{r^3}(1 - 3\cos^2\theta) + P\cos\theta\cos\varphi\right] & -Q\sin\varphi\cos\varphi & 0 & -\left[\frac{D}{mv} + Q\cos^2\varphi\right]
\end{bmatrix}
$$

The matrix F(t) is determined by the nominal trajectory and the column G(t) by the partial derivatives of the disturbing potential w evaluated on the nominal trajectory. A comparison of this matrix differential equation with the differential equations (7) for the portion of the flight prior to reentry shows that the matrix differential equation may be employed for the entire flight if P, Q, and D are taken as zero for the portion of the flight prior to reentry.

The solution of the differential equation (9) is

$$x(t) = X(t,t_0)x(t_0) + \int_{t_0}^{t} X(t,\tau)G(\tau)d\tau ,$$

where $x(t_0)$ is the value of x at the time t_0 of launch and where the 6 X 6 matrix $X(t,\tau)$ is the solution of the homogeneous matrix differential equation

$$\frac{dX}{dt} = F(t)X$$

subject to the initial condition

$$X(\tau,\tau) = I = 6 \times 6 \text{ identity matrix.}$$

The time t_0 of launch will in practice be taken as zero but is retained as t_0 in some of the equations for purposes of clarity. We will be interested in the value of $x(t)$ at the time $t = T$ of impact of the nominal missile, that is, in

$$x(T) = X(T,t_0)x(t_0) + \int_{t_0}^{T} X(T,t)G(t)dt\,. \tag{10}$$

It can be shown that $X(T,t) = Y^T(t,T)$, where the superscript T denotes matrix transpose and $Y(t,T)$ is the solution of the "adjoint" matrix differential equation

$$\frac{dY}{dt} = -F^T(t)Y \tag{11}$$

subject to the "initial" condition

$$Y(T,T) = I\,.$$

For use in Equation (10), $X(T,t)$ is required not only for $t = t_0$ but, to evaluate the integral, for all values of t in the range $t_0 \leqslant t \leqslant T$. To obtain these matrices, the homogeneous differential equation which is actually solved is that for $Y(t,T)$, starting at $t = T$ with the identity matrix and integrating backwards in time to $t = t_0$. This gives $Y(t,T)$ over the range $T \geqslant t \geqslant t_0$ and hence $X(T,t) = Y^T(t,T)$ over the range $t_0 \leqslant t \leqslant T$. An examination of the form of $F(t)$ shows that $X(T,t)$ must have zero elements in the positions indicated in the expression

$$X(T,t) = \begin{bmatrix} X_{11} & 0 & X_{13} & X_{14} & 0 & X_{16} \\ 0 & X_{22} & 0 & 0 & X_{25} & 0 \\ X_{31} & 0 & X_{33} & X_{34} & 0 & X_{36} \\ X_{41} & 0 & X_{43} & X_{44} & 0 & X_{46} \\ 0 & X_{52} & 0 & 0 & X_{55} & 0 \\ X_{61} & 0 & X_{63} & X_{64} & 0 & X_{66} \end{bmatrix}$$

IV. IMPACT ERRORS

The column x(T) is formed from the differences $\xi(T)$, $\eta(T)$, $\zeta(T)$, $\dot{\xi}(T)$, $\dot{\eta}(T)$, $\dot{\zeta}(T)$ between the position and velocity components of the actual missile and those of the nominal missile at the instant of impact of the nominal missile. Figure 2 illustrates the nominal trajectory

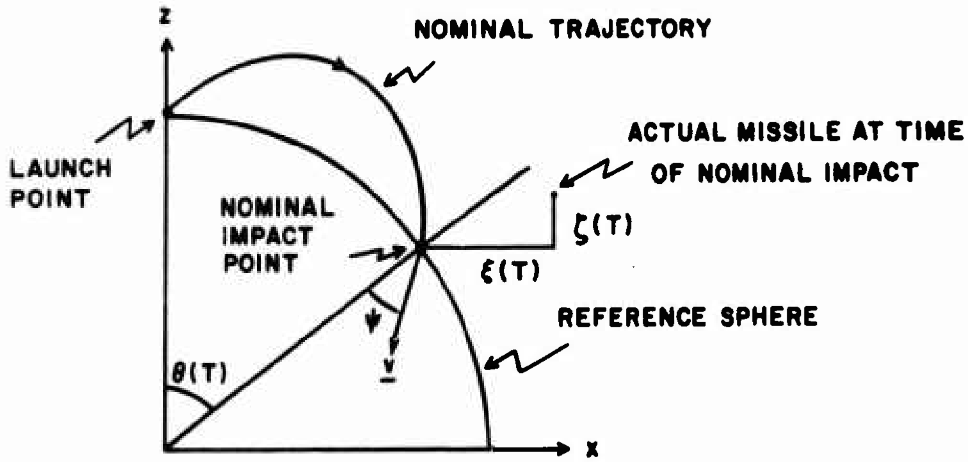

FIGURE 2

in the zx-plane. The nominal missile strikes the target on the reference sphere at range angle $\theta(T)$. At this instant the actual missile is displaced from the nominal missile by a component $\xi(T)$ in the x-direction, a component $\zeta(T)$ in the z-direction, and a component $\eta(T)$ (not shown in the diagram) in the y-direction. During the additional time to impact of the actual missile it will move from this position. Since only first order effects are being considered in the analysis, the velocity of the actual missile during this small time interval is regarded as being equal to the impact velocity of the nominal missile. This velocity $\underline{v}$ forms an angle ψ with the downward vertical at the target.

Since $\underline{v}$ lies in the zx-plane, the actual missile will have no motion in the y-direction during the small time increment under consideration and the cross-range component of impact error will be $\delta C = \eta(T)$, taken as positive to the left when viewed from the launch point. At the nominal impact time, the actual missile is

above the target a distance $\xi(T) \sin \theta(T) + \zeta(T) \cos \theta(T)$ and is down range from the target a distance $\xi(T) \cos \theta(T) - \zeta(T) \sin \theta(T)$. Since the vertical component of velocity of the missile is $v \cos \psi$, it will impact at a time $[\xi(T) \sin \theta(T) + \zeta(T) \cos \theta(T)]/(v \cos \psi)$ later than the nominal missile. During this time its down-range component of velocity is $v \sin \psi$ and hence it will travel a distance $[\xi(T) \sin \theta(T) + \zeta(T) \cos \theta(T)] \tan \psi$ in the down-range direction. Adding this to the down-range distance from the target at the nominal impact time, the down-range component of impact error is

$$
\begin{aligned}
\delta D &= \xi(T) \cos \theta(T) - \zeta(T) \sin \theta(T) \\
&\quad + [\xi(T) \sin \theta(T) + \zeta(T) \cos \theta(T)] \tan \psi \\
&= a\xi(T) + b\zeta(T)
\end{aligned}
$$

where

$$
\begin{aligned}
a &= \tan \psi \sin \theta(T) + \cos \theta(T) \\
b &= \tan \psi \cos \theta(T) - \sin \theta(T) .
\end{aligned}
$$

The angles $\theta(T)$ and ψ can be expressed in terms of the x and z components of position and velocity of the nominal missile at time $t = T$ and a and b expressed in the forms

$$
\begin{aligned}
a &= + \frac{R}{x\dot{x} + z\dot{z}} \dot{z} \\
b &= - \frac{R}{x\dot{x} + z\dot{z}} \dot{x}.
\end{aligned}
\tag{12}
$$

Because the anomalous field and the geoidal height have been defined in terms of the complex surface spherical harmonics $Y_n^m(\theta,\lambda)$ with the complex coefficients α_n^m, it becomes convenient to combine the impact error components into a complex impact error

$$\begin{aligned}\rho &= \delta D + i\delta C \\ &= a\xi(T) + i\eta(T) + b\zeta(T) \\ &= [a\ \ i\ \ b] \begin{bmatrix}\xi(T)\\ \eta(T)\\ \zeta(T)\end{bmatrix} .\end{aligned}$$

From this equation it is seen that only the first three elements of x(T) are required and hence only the first three rows of the matrix X(T,t). Furthermore, under the assumption that the ship has remained stationary for a considerable period before the missile is launched, and the further assumption that the ship's inertial navigation system operates in a "damped" mode involving the use of non-inertial horizontal velocity data and height (or depth) data, the velocity outputs of the navigation system will be zero so that $\dot{\xi}(t_0) = \dot{\eta}(t_0) = \dot{\zeta}(t_0) = 0$. Therefore, in the product $X(T,t_0)\ x(t_0)$ in Equation (10), only the first three columns of X(T,t) are needed. In the integrand function X(T,t) G(t) of the same equation, the first three elements of G(t) are zero so only the last three columns of X(T,t) are required. Making use of these facts it is found that ρ can be written in the form

$$\begin{aligned}\rho = &\left[aX_{11} + bX_{31}\ \ X_{22}\ \ aX_{13} + bX_{33}\right] \begin{bmatrix}\xi_0\\ i\eta_0\\ \zeta_0\end{bmatrix} \\ &+ \int_{t_0}^{T} \left[aX_{14} + bX_{34}\ \ X_{25}\ \ aX_{16} + bX_{36}\right] \begin{bmatrix}\frac{\partial w}{\partial x}\\ i\frac{\partial w}{\partial y}\\ \frac{\partial w}{\partial z}\end{bmatrix} dt\end{aligned} \tag{13}$$

where, now, $\xi_0 = \xi(t_0)$, etc. The matrix elements X_{ij} in the first row stand for $X_{ij}(T,t_0)$, which are constants for a particular nominal trajectory. The matrix elements which appear in the row under the integral sign stand for $X_{ij}(T,t)$, which are functions of time along the nominal trajectory.

The two rows (1 X 3 matrices) appearing in Equation (13) involve a,b, and the matrix elements X_{ij} and depend only on the nominal trajectory. The two columns depend on the anomalous field. The first column; whose elements are ξ_0, $i\eta_0$, ζ_0; represents the initial position errors due to the anomalous field and is independent of the trajectory. When multiplied by the trajectory-dependent row $[aX_{11} + bX_{31} X_{22} aX_{13} + bX_{33}]$, the contribution to ρ of the initial position errors

(i.e., ship's navigation errors) is obtained. The second column; whose elements are $\partial w/\partial w$, $i\partial w/\partial y$, $\partial w/\partial z$; depends on the anomalous field (since w is the disturbing potential) and also depends on the nominal trajectory, since the partial derivatives are to be evaluated on this trajectory. This column is then multiplied by the trajectory-dependent row $[aX_{14} + bX_{34} \;\; X_{25} \;\; aX_{16} + bX_{36}]$ and the product is integrated. This integral, being independent of the initial position errors, represents the in-flight contribution to the complex impact error ρ. In the next two sections the navigation and in-flight contributions to ρ are evaluated separately.

V. NAVIGATION CONTRIBUTION TO IMPACT ERROR

The launch point of the nominal missile is the north pole of the reference sphere and the ship attempts to bring itself to this position by using the outputs of its inertial navigation system. The initial position errors ξ_0, η_0, ζ_0 are the component displacements of the ship from the north pole of the reference sphere when the erroneous outputs of the navigation system indicate that it is at the north pole of the reference sphere. Since the navigation system contains no information on the geoidal height function, the actual position of the ship will be on the geoid instead of on the reference sphere. Since the navigation system also contains no information on the vertical deflections and operates in a damped mode, the vertical indication of the navigation system will be the plumb line vertical which is normal to the geoid. It has also been assumed that the inertial navigator employs perfect gyroscopes. It can therefore be supposed that it possesses axes in the x,y,z directions whose orientation has been accurately established prior to the mission by an in-port alignment procedure and that this orientation is maintained throughout the mission by the perfect gyros. Since the ship is believed to be at the north pole of the reference sphere, its actual position will therefore be at such a place on the geoid that the normal to the geoid is parallel to the z-axis.

From Equations (1) the parametric equations of the geoid are

$$
\begin{aligned}
x &= (R + N) \sin\theta \cos\lambda \\
y &= (R + N) \sin\theta \sin\lambda \\
z &= (R + N) \cos\theta
\end{aligned}
\qquad (14)
$$

where $N = N(\theta,\lambda)$ is given in terms of θ, λ, and the α_n^m by Equation (5). If θ and λ are given small increments $d\theta$ and $d\lambda$, the increment of z will be

$$dz = \left[\frac{\partial N}{\partial\theta}\cos\theta - (R + N)\sin\theta\right]d\theta + \frac{\partial N}{\partial\lambda}\cos\theta d\lambda.$$

If the increments $d\theta$ and $d\lambda$ are from such values of θ and λ that the normal to the geoid is parallel to the z-axis, then $dz = 0$ and hence the coefficients of $d\theta$ and $d\lambda$ are zero, or

$$\frac{\partial N}{\partial \theta}\cos\theta = (R + N)\sin\theta$$
$$\frac{\partial N}{\partial \lambda}\cos\theta = 0 . \tag{15}$$

These are two simultaneous equations for θ and λ. If they can be solved and the solution values substituted in Equations (14), the coordinates of the ship will be obtained and the initial position errors will be (since the intended coordinates are 0,0,R)

$$\begin{aligned} \xi_0 &= (R + N)\sin\theta\cos\lambda \\ \eta_0 &= (R + N)\sin\theta\sin\lambda \\ \zeta_0 &= (R + N)\cos\theta - R . \end{aligned} \tag{16}$$

It is possible that Equations (15) have many solutions for θ and λ and that some of these solutions might have large values of θ and represent ship's positions far from the north pole. This statement corresponds to the fact that it is possible conceptually, although it is probably not actually the case, that there may be distinct points on the geoid having the same astronomic coordinates, i.e., distinct points where the plumb lines are parallel. However, if the α_n^m are all zero the geoid coincides with the reference sphere and the only solution (except for the south pole) is $\theta = 0$. Hence, if the α_n^m are small there must be a solution having a small value of θ and this solution will be sought as the one appropriate to the problem.

Since θ vanishes with the α_n^m and powers and products of these small quantities are being neglected, the approximations $\sin\theta = \theta$ and $\cos\theta = 1$ can be made in Equations (15) to obtain

$$\frac{\partial N}{\partial \theta} = (R + N)\theta$$
$$\frac{\partial N}{\partial \lambda} = 0 . \tag{17}$$

The functions N, $\partial N/\partial\theta$, $\partial N/\partial\lambda$ appearing in these equations can be expanded as power series in θ (λ being regarded as a parameter) of the forms

$$N = N|_0 + \left.\frac{\partial N}{\partial\theta}\right|_0 \theta$$

$$\frac{\partial N}{\partial\theta} = \left.\frac{\partial N}{\partial\theta}\right|_0 + \left.\frac{\partial^2 N}{\partial\theta^2}\right|_0 \theta$$

$$\frac{\partial N}{\partial\lambda} = \left.\frac{\partial N}{\partial\lambda}\right|_0 + \left.\frac{\partial^2 N}{\partial\theta\partial\lambda}\right|_0 \theta \, .$$

in which terms involving powers of θ higher than the first have been neglected. The bar on the right indicates an evaluation at $\theta = 0$ which may, however, depend on λ. Substituting these expansions in Equations (17) and again omitting terms involving θ^2, one obtains

$$\left.\frac{\partial N}{\partial\theta}\right|_0 + \left.\frac{\partial^2 N}{\partial\theta^2}\right|_0 \theta = (R + N|_0)\theta$$

(18)

$$\left.\frac{\partial N}{\partial\lambda}\right|_0 + \left.\frac{\partial^2 N}{\partial\theta\partial\lambda}\right|_0 \theta = 0 \, .$$

To obtain N and its partial derivatives evaluated at $\theta = 0$, Equation (5) is first rewritten with the substitution of the expressions (2) for the $Y_n^m(\theta,\lambda)$:

$$N = R \sum_n \sum_m \alpha_n^m (-1)^m \sqrt{\frac{2n+1}{4\pi}\,\frac{(n-m)!}{(n+m)!}}\, P_n^m(\cos\theta) e^{im\lambda} \tag{19}$$

Differentiation with respect to θ yields

$$\frac{\partial N}{\partial\theta} = R \sum_n \sum_m \alpha_n^m (-1)^m \sqrt{\frac{2n+1}{4\pi}\,\frac{(n-m)!}{(n+m)!}}\, \frac{dP_n^m(\cos\theta)}{d\theta} e^{im\lambda}$$

and, using

$$\sin\theta \frac{dP_n^m(\cos\theta)}{d(\cos\theta)} = \frac{1}{2} P_n^{m+1}(\cos\theta) - \frac{1}{2}(n+m)(n-m+1)P_n^{m-1}(\cos\theta), \tag{20}$$

this becomes

$$\frac{\partial N}{\partial \theta} = -\frac{R}{2}\sum_n \sum_m \alpha_n^m (-1)^m \sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}}\left[P_n^{m+1} - (n+m)(n-m+1)P_n^{m-1}\right] e^{im\lambda} .$$

In this and some later equations the argument of the P_n^m is always to be understood as $\cos\theta$ when it is not exhibited explicitly. Differentiating (19) with respect to λ gives

$$\frac{\partial N}{\partial \lambda} = iR \sum_n \sum_m \alpha_n^m (-1)^m m \sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}} P_n^m e^{im\lambda}$$

and the second partial derivative

$$\frac{\partial^2 N}{\partial\theta\partial\lambda} = -i\frac{R}{2}\sum_n \sum_m \alpha_n^m (-1)^m m \sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}}\left[P_n^{m+1} - (n+m)(n-m+1)P_n^{m-1}\right] e^{im\lambda} .$$

To evaluate these expressions at $\theta = 0$, use is made of the relation, from (2),

$$P_n^m(\cos 0) = \frac{(-1)^m Y_n^m(0,\lambda)e^{-im\lambda}}{\sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}}}$$

and the fact that

$$Y_n^m(0,\lambda) = \sqrt{\frac{2n+1}{4\pi}}\,\delta_0^m$$

where δ_0^m is the Kronecker delta. With the abbreviation $A_n = \sqrt{\frac{n(n+1)(2n+1)}{4\pi}}$ the results are

$$
\begin{aligned}
N|_0 &= R\sum_n \sqrt{\frac{2n+1}{4\pi}}\,\alpha_n^o \\
\left.\frac{\partial N}{\partial \theta}\right|_0 &= -\frac{R}{2}\left[e^{i\lambda}\sum_n A_n\alpha_n^1 - e^{-i\lambda}\sum_n A_n\alpha_n^{-1}\right] \\
\left.\frac{\partial N}{\partial \lambda}\right|_0 &= 0 \\
\left.\frac{\partial^2 N}{\partial \theta\partial \lambda}\right|_0 &= -i\frac{R}{2}\left[e^{i\lambda}\sum_n A_n\alpha_n^1 + e^{-i\lambda}\sum_n A_n\alpha_n^{-1}\right].
\end{aligned}
\tag{21}
$$

The second partial derivative $\partial^2 N/\partial\theta^2|_0$ has not been evaluated because it is multiplied by θ in the first of Equations (18) and this product contains no first order terms in the α_n^m. This product and also the product $N|_0\theta$ in the same equation are therefore omitted. With these approximations, the substitution of the values (21) in Equations (18) yields

$$
\begin{aligned}
-\frac{R}{2}\left[e^{i\lambda}\sum_n A_n\alpha_n^1 - e^{-i\lambda}\sum_n A_n\alpha_n^{-1}\right] &= R\theta \\
-i\frac{R}{2}\left[e^{i\lambda}\sum_n A_n\alpha_n^1 + e^{-i\lambda}\sum_n A_n\alpha_n^{-1}\right]\theta &= 0.
\end{aligned}
\tag{22}
$$

We are seeking a value of $\theta \neq 0$ which satisfies these equations and hence

$$
e^{i\lambda}\sum_n A_n\alpha_n^1 + e^{-i\lambda}\sum_n A_n\alpha_n^{-1} = 0 \tag{23}
$$

from which λ could be determined. With this value of λ the first of Equations (22) would give the value of θ. These two values could be substituted in the first two of Equations (16) to obtain ξ_0 and η_0. However, these equations can be combined into the form

$$\xi_0 + i\eta_0 = (R + N) \sin \theta (\cos \lambda + i \sin \lambda) = (R + N) \sin \theta \, e^{i\lambda}$$

or, to the desired degree of approximation,

$$\xi_0 + i\eta_0 = R \theta \, e^{i\lambda} .$$

Using the first of Equations (22),

$$\begin{aligned} \xi_0 + i\eta_0 &= -\frac{R}{2}\left[e^{i\lambda} \sum_n A_n \alpha_n^1 - e^{-i\lambda} \sum_n A_n \alpha_n^{-1}\right] e^{i\lambda} \\ &= -\frac{R}{2}\left[e^{2i\lambda} \sum_n A_n \alpha_n^1 - \sum_n A_n \alpha_n^{-1}\right] . \end{aligned}$$

But λ satisfies (23), which can be written

$$e^{2i\lambda} \sum_n A_n \alpha_n^1 = -\sum_n A_n \alpha_n^{-1}$$

and hence

$$\begin{aligned} \xi_0 + i\eta_0 &= R \sum_n A_n \alpha_n^{-1} \\ &= -R \sum_n A_n \overline{\alpha_n^1} . \end{aligned}$$

The conjugate of this expression is

$$\xi_0 - i\eta_0 = -R \sum_n A_n \alpha_n^1 .$$

The last two equations can be solved for ξ_0 and $i\eta_0$ to obtain

$$\xi_0 = -\frac{R}{2}\sum_n A_n(\alpha_n^1 + \overline{\alpha_n^1}) = -\frac{R}{2}\sum_n A_n(\alpha_n^1 - \alpha_n^{-1})$$

$$i\eta_0 = +\frac{R}{2}\sum_n A_n(\alpha_n^1 - \overline{\alpha_n^1}) = +\frac{R}{2}\sum_n A_n(\alpha_n^1 + \alpha_n^{-1}) \,. \tag{24}$$

We also need the component ζ_0 of initial position error which, from (16), is

$$\zeta_0 = (R + N)\cos\theta - R$$

or, approximately,

$$\zeta_0 = N|_0 = R\sum_n \sqrt{\frac{2n+1}{4\pi}}\,\alpha_n^o \,. \tag{25}$$

For use in later developments it is desirable to express Equations (24) and (25) as double sums (over m as well as n) by introducing the Kronecker deltas. Recalling the definition of A_n, these equations then become

$$\begin{aligned}
\xi_0 &= -\frac{R}{2}\sum_n \sum_m \alpha_n^m \sqrt{\frac{2n+1}{4\pi}}\sqrt{n(n+1)}\left(\delta_1^m - \delta_{-1}^m\right) \\
i\eta_0 &= +\frac{R}{2}\sum_n \sum_m \alpha_n^m \sqrt{\frac{2n+1}{4\pi}}\sqrt{n(n+1)}\left(\delta_1^m + \delta_{-1}^m\right) \\
\zeta_0 &= +R\sum_n \sum_m \alpha_n^m \sqrt{\frac{2n+1}{4\pi}}\,\delta_0^m
\end{aligned} \tag{26}$$

We now return to Equation (13) and define trajectory-dependent constants

$$
\begin{aligned}
g_1 &= aX_{11}(T,t_0) + bX_{31}(T,t_0) \\
g_2 &= X_{22}(T,t_0) \\
g_3 &= aX_{13}(T,t_0) + bX_{33}(T,t_0)\,.
\end{aligned}
\tag{27}
$$

Using these constants, the navigation contribution to the complex impact error can be written first as

$$
\rho(\mathrm{Nav}) = g_1 \xi_0 + g_2 i\eta_0 + g_3 \zeta_0
$$

and then, by introducing the values (26), as

$$
\begin{aligned}
\rho(\mathrm{Nav}) = \sum_n \sum_m \alpha_n^m \sqrt{\frac{2n+1}{4\pi}} \Bigg[&- Rg_1 \frac{\sqrt{n(n+1)}}{2} (\delta_1^m - \delta_{-1}^m) \\
&+ Rg_2 \frac{\sqrt{n(n+1)}}{2} (\delta_1^m + \delta_{-1}^m) + Rg_3 \delta_0^m \Bigg].
\end{aligned}
\tag{28}
$$

VI. IN-FLIGHT CONTRIBUTION TO IMPACT ERROR

This contribution to the complex impact error arises from the integral expression in Equation (13). It has been explained how the 1 X 3 matrix in the integrand can be obtained, as a function of t, by integration of the adjoint Equation (11). We now consider the column involving the partial derivatives of the disturbing potential w. These partial derivatives are related to the partial derivatives of the spherical coordinate form W of the disturbing potential by the transformation

$$\begin{bmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial \theta}{\partial x} & \frac{\partial \lambda}{\partial x} & \frac{\partial r}{\partial x} \\ \frac{\partial \theta}{\partial y} & \frac{\partial \lambda}{\partial y} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial z} & \frac{\partial \lambda}{\partial z} & \frac{\partial r}{\partial z} \end{bmatrix} \begin{bmatrix} \frac{\partial W}{\partial \theta} \\ \frac{\partial W}{\partial \lambda} \\ \frac{\partial W}{\partial r} \end{bmatrix} .$$

The 3 X 3 matrix is evaluated by making use of the relations (1) between rectangular and spherical coordinates and remembering that. because the partial derivatives are to be evaluated on the nominal trajectory for which $\lambda = 0$, the partial derivatives in the 3 X 3 matrix can be evaluated for $\lambda = 0$. Also introducing the factor i we obtain

$$\begin{bmatrix} \frac{\partial w}{\partial x} \\ i\,\frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial z} \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & +\sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \frac{1}{r}\,\frac{\partial W}{\partial \theta} \\ \frac{i}{r\sin\theta}\,\frac{\partial W}{\partial \lambda} \\ \frac{\partial W}{\partial r} \end{bmatrix} .$$

The integrand involved in the complex impact error therefore becomes

$$[aX_{14} + bX_{34} \quad X_{25} \quad aX_{16} + bX_{36}] \begin{bmatrix} \cos\theta & 0 & +\sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \frac{1}{r}\,\frac{\partial W}{\partial \theta} \\ \frac{i}{r\sin\theta}\,\frac{\partial W}{\partial \lambda} \\ \frac{\partial W}{\partial r} \end{bmatrix} .$$

By introducing the functions

$$
\begin{aligned}
f_1(t) &= (aX_{14} + bX_{34})\cos\theta - (aX_{16} + bX_{36})\sin\theta \\
f_2(t) &= X_{25} \qquad\qquad (29) \\
f_3(t) &= (aX_{14} + bX_{34})\sin\theta + (aX_{16} + bX_{36})\cos\theta
\end{aligned}
$$

the integrand may be written as

$$
f_1(t)\frac{1}{r}\frac{\partial W}{\partial\theta} + f_2(t)\frac{i}{r\sin\theta}\frac{\partial W}{\partial\lambda} + f_3(t)\frac{\partial W}{\partial r}\ .
$$

The functions $f_i(t)$ depend on t not only through the factors $\sin\theta$ and $\cos\theta$ but also because the matrix elements $X_{ij} = X_{ij}(T,t)$.

The partial derivatives of W are found by using the expression (4) for $W(r,\theta,\lambda)$ and the definitions of $Y_n^m(\theta,\lambda)$ and $P_n^m(\cos\theta)$ in (2) and (3). In addition, formula (20) and the recursion formula

$$
P_n^{m+1}(\cos\theta) - 2m\cot\theta P_n^m(\cos\theta) + (n+m)(n-m+1)P_n^{m-1}(\cos\theta) = 0
$$

are required. After forming the partial derivatives, λ is set equal to zero, since the derivatives are to be evaluated on the nominal trajectory. The results of these operations are

$$
\frac{1}{r}\frac{\partial W}{\partial\theta} = +\frac{\mu}{r^2}\sum_n\left(\frac{R}{r}\right)^n\sum_m \alpha_n^m(-1)^m\sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}}\left(m\cot\theta\cdot P_n^m - P_n^{m+1}\right)
$$

$$
\frac{i}{r\sin\theta}\frac{\partial W}{\partial\lambda} = -\frac{\mu}{r^2}\sum_n\left(\frac{R}{r}\right)^n\sum_m \alpha_n^m(-1)^m\sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}}\ m\csc\theta\cdot P_n^m
$$

$$
\frac{\partial W}{\partial r} = -\frac{\mu}{r^2}\sum_n(n+1)\left(\frac{R}{r}\right)^n\sum_m \alpha_n^m(-1)^m\sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}}P_n^m\ .
$$

Substituting these expansions in the integrand function and interchanging the orders of integration and summation, the in-flight contribution to the complex impact error becomes

$$\rho(I - F) = \sum_n \sum_m \alpha_n^m (-1)^m \mu R^n \sqrt{\frac{2n+1}{4\pi} \frac{(n-m)!}{(n+m)!}} \int_{t_0}^{T} \frac{1}{r^{n+2}} \Big[f_1 \Big(m \cot\theta P_n^m - P_n^{m+1} \Big)$$

$$- f_2 m \csc\theta \, P_n^m - f_3 (n+1) P_n^m \Big] \, dt \, .$$

With the substitutions

$$I_n^m = \int_{t_0}^{T} \frac{1}{r^{n+2}} \Big[f_1(t)(m \cot\theta \cdot P_n^m - P_n^{m+1}) - f_3(t)(n+1) P_n^m \Big] \, dt$$

$$J_n^m = \int_{t_0}^{T} \frac{1}{r^{n+2}} f_2(t) \, m \csc\theta \cdot P_n^m \, dt \qquad (30)$$

this becomes

$$\rho(I - F) = \sum_n \sum_m \alpha_n^m (-1)^m \mu R^n \sqrt{\frac{2n+1}{4\pi} \frac{(n-m)!}{(n+m)!}} \, (I_n^m - J_n^m) \, . \qquad (31)$$

VII. THE TOTAL COMPLEX IMPACT ERROR

The total impact error ρ in Equation (13) is the sum of the navigation contribution $\rho(\mathrm{Nav})$ in (28) and the in-flight contribution $\rho(\mathrm{I-F})$ in (31). This sum can be written in the form

$$\rho = \sum_n \sum_m \alpha_n^m c_n^m \tag{32}$$

where

$$c_n^m = \sqrt{\frac{2n+1}{4\pi}} \left[- Rg_1 \frac{\sqrt{n(n+1)}}{2} (\delta_1^m - \delta_{-1}^m) + Rg_2 \frac{\sqrt{n(n+1)}}{2} (\delta_1^m + \delta_{-1}^m) + Rg_3 \delta_0^m \right.$$
$$\left. + (-1)^m \mu R^n \sqrt{\frac{(n-m)!}{(n+m)!}} \left(I_n^m - J_n^m \right) \right] . \tag{33}$$

For use in some later operations it should be noted that the c_n^m are real.

To summarize the development thus far, suppose that the nominal missile is launched from the north pole of the reference sphere and strikes a target in the Greenwich Meridian. The radius vector r and range angle θ of this missile are known functions $r = r(t)$ and $\theta = \theta(t)$ of the time, with impact occurring at time $t = T$. By integrating the adjoint Equation (11) corresponding to this trajectory, the constants g_1, g_2, g_3 defined by Equations (27) and the functions $f_1(t)$, $f_2(t)$, $f_3(t)$ defined by Equations (29) are determined. The integrals I_n^m and J_n^m defined by Equations (30) can therefore be evaluated and the quantities c_n^m computed. The complex impact error of the actual missile is then given by Equation (32), where the α_n^m are the coefficients in the expansion (5) of the geoidal height function. The impact error thus derived includes the effect of the initial position error resulting from the action of the anomalous field on the ship's inertial navigation system and the in-flight effect due to the action of the anomalous field on the flight of the missile.

VIII. ARBITRARY LAUNCH POSITION AND TARGET AZIMUTH

It will next be explained how the previous results can be applied to the determination of the impact error of a missile which is launched from an arbitrary position on the earth toward a target at an arbitrary azimuth. The principle which is involved in this extension of the results depends on the fact that in the case already considered the nominal launch position was required to be at the north pole of the reference sphere and the target in the meridian of Greenwich only because the expansion of the geoidal height function was made in the terms of the functions $Y_n^m(\theta,\lambda)$ of colatitude θ (measured from the north pole) and longitude λ (measured from the Greenwich Meridian), and the coefficients α_n^m involved in the expansion were based on the use of these particular functions. If we wish to consider a launch point other than the north pole, we can regard the launch point as the fictitious "north pole" of a new coordinate system and the plane of the nominal trajectory as the fictitious "Greenwich Meridian" of the new coordinate system. In this new coordinate system, the geoidal height would naturally be expressed as a function $N' = N'(\theta',\lambda')$ of "colatitude" θ' and "longitude" λ', and expanded in terms of the functions $Y_n^m(\theta',\lambda')$. The coefficients in this expansion would be certain constants β_n^m instead of the α_n^m in the original expansion. If these constants can be determined, the complex impact error for the new situation can be computed, by analogy with Equation (32), from the equation

$$\rho = \sum_n \sum_m \beta_n^m c_n^m , \tag{34}$$

where the constants c_n^m have the same values that they have for the nominal trajectory originating at the north pole and lying in the Greenwich Meridian.

To carry out this idea, let α represent the longitude of the new launch point, β its colatitude, and γ the azimuth of the target, measured CCW from the south. The relation of the new coordinate system (x', y', z' or θ', λ') to that originally used (x,y,z or θ,λ) is indicated in Figure 3, while Figure 4 shows how the "colatitude" θ' and "longitude" λ' of an arbitrary point on the reference sphere are related to the true colatitude θ and true longitude λ of the same point. The surface spherical harmonics $Y_n^m(\theta',\lambda')$ in the new coordinate system are related to those in the old system (M. E. Rose, ***Elementary Theory of Angular Momentum***, Equation (4.28a)) by the equation

RELATION BETWEEN X,Y,Z AXES AND X', Y', Z' AXES

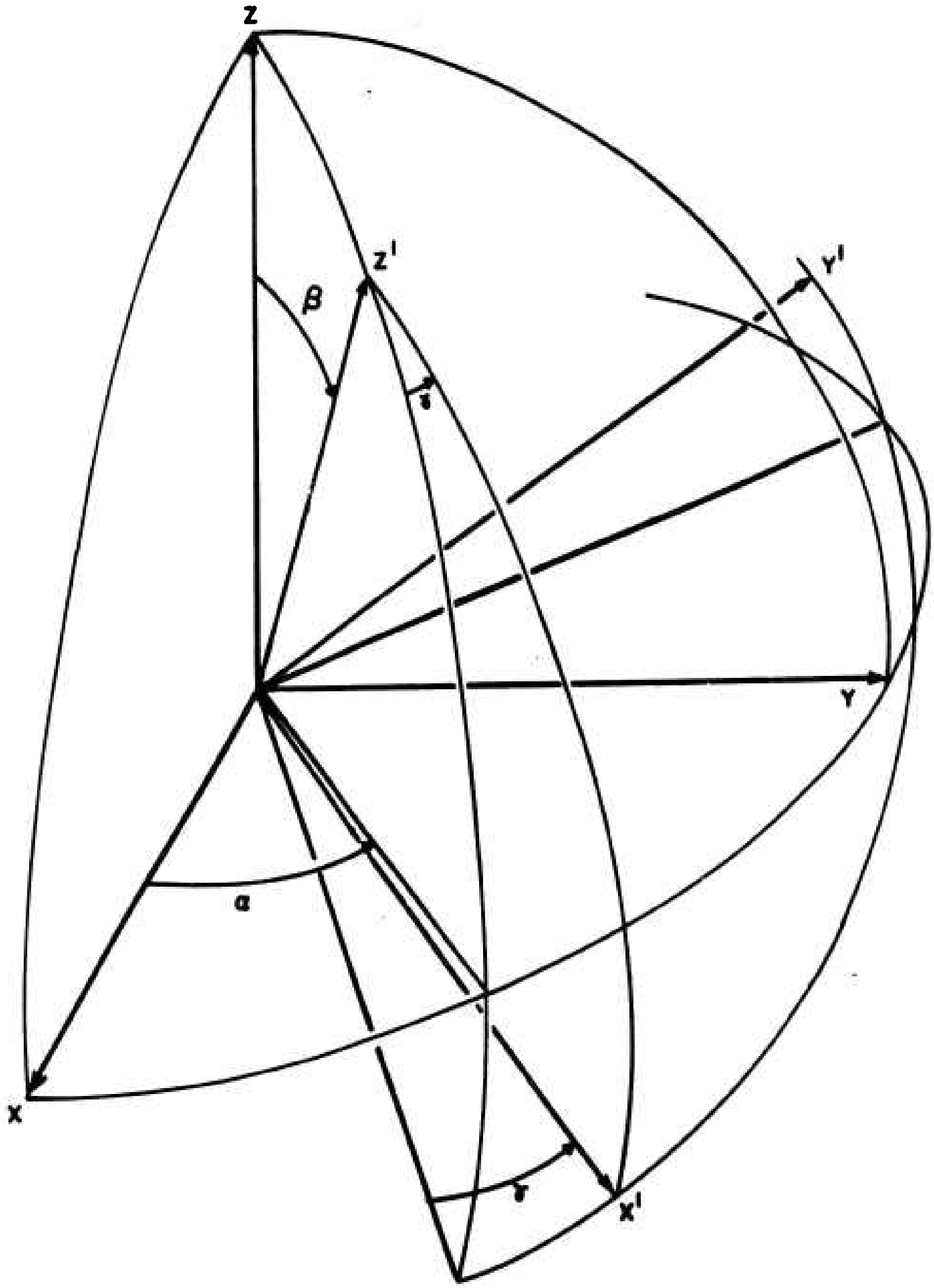

FIGURE 3

RELATION BETWEEN θ, λ COORDINATES AND θ', λ' COORDINATES

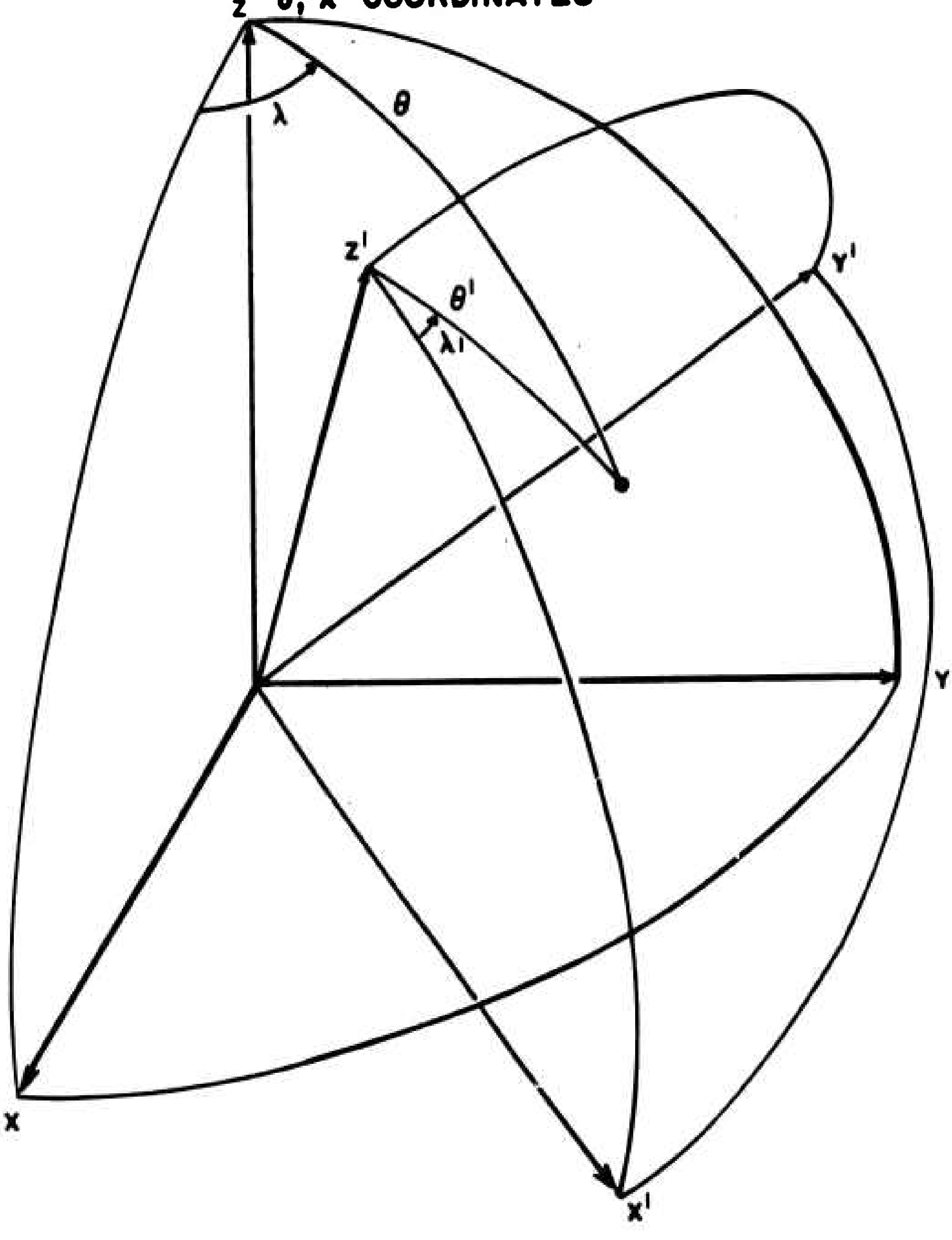

FIGURE 4

$$Y_n^m(\theta',\lambda') = \sum_{k=-n}^{+n} D_{km}^n(\alpha,\beta,\gamma) Y_n^k(\theta,\lambda), \tag{35}$$

where the functions D_{km}^n have a simple dependence on α and γ but a rather complicated dependence on β. The actual form of these functions will not, however, be needed. The inverse relationship is

$$Y_n^m(\theta,\lambda) = \sum_k D_{km}^n(-\gamma,-\beta,-\alpha) Y_n^k(\theta',\lambda')$$

which, by virtue of Rose's Equation (4.21), may be written

$$Y_n^m(\theta,\lambda) = \sum_k \overline{D_{mk}^n(\alpha,\beta,\gamma)}\, Y_n^k(\theta',\lambda') .$$

The geoidal height expansion (5) can therefore by written as

$$\begin{aligned} N(\theta,\lambda) &= R\sum_n \sum_m \alpha_n^m \sum_k \overline{D_{mk}^n(\alpha,\beta,\gamma)}\, Y_n^k(\theta',\lambda') \\ &= R \sum_n \sum_m \left[\sum_k \alpha_n^k \overline{D_{km}^n(\alpha,\beta,\gamma)}\; Y_n^m(\theta',\lambda')\right]. \end{aligned}$$

This is an expansion of the geoidal height in the form

$$N'(\theta',\lambda') = R \sum_n \sum_m \beta_n^m Y_n^m(\theta',\lambda')$$

and hence

$$\beta_n^m = \beta_n^m(\alpha,\beta,\gamma) = \sum_k \alpha_n^k \overline{D_{km}^n(\alpha,\beta,\gamma)} .$$

The complex impact error for the case now under consideration can therefore be written

$$\rho(\alpha,\beta,\gamma) = \sum_n \sum_m \sum_k c_n^m \alpha_n^k \overline{D_{km}^n(\alpha,\beta,\gamma)} . \tag{36}$$

IX. STATISTICAL CONSIDERATIONS

Up to this point, the development of an expression for the impact error has been entirely deterministic. We might be given at the outset a particular nominal trajectory and could compute the values of the c_n^m given by Equation (33) without regard to the launch position and target azimuth. We would also be given the values of the α_n^m in the expansion of the geoidal height. A particular launch point would then be specified by the values of α and β and a particular target azimuth by the value of γ. Using the well-defined functions $D_{km}^n(\alpha, \beta, \gamma)$ whose explicit form has not, however, been displayed, the coefficients β_n^m could be computed. Finally, the complex impact error would be computed from Equation (34).

Although all of these computations might actually be carried out, we are in practice often interested in obtaining only a statistical description of the impact errors for some ensemble of launch conditions. The statistical description which would be most useful would be to consider a particular target and to describe statistically the impact errors of missiles launched toward this target from various points of some limited launch area. If the earth's gravitational field were known, but had deliberately been neglected, this problem could be solved by a Monte Carlo method in which a numerical statistical analysis would be made of the impact errors, computed as just outlined, of missiles launched from a large number of randomly selected points within the launch area. Such a computation would be extremely time-consuming as a result of the large number of nominal trajectories (many sets of the c_n^m) and the large number of launch points (many sets of the β_n^m) that would have to be considered to obtain an adequate sample of the impact errors. Furthermore, the results obtained would apply to the case where the known field had deliberately been neglected, a situation in which the impact errors could have been avoided by taking that field into account. By considering a different ensemble of launch conditions, however, it becomes possible to obtain a statistical description of the impact errors by a method which is largely analytical and involves numerical computation to a much lesser extent. Of greater importance, this method is applicable to the situation where the anomalous field has not been taken into account because it is not completely known.

The situation considered is one in which a particular nominal trajectory is chosen; defined, for example, by range and time of flight. Then it is supposed that missiles are launched from positions uniformly distributed over the surface of the earth and that, for each such launch point, missiles are launched at targets having a range from the launch point equal to the range of the nominal trajectory and being uniformly distributed in azimuth. A statistical description of the impact errors in such a situation is the goal of the remainder of this report.

The first statistic to be computed is the expected or mean value $\tilde{\rho}$ of the complex impact error $\rho(\alpha,\beta,\gamma)$. This has the value

$$\tilde{\rho} = \frac{1}{8\pi^2} \int_{\alpha=0}^{2\pi} \int_{\beta=0}^{\pi} \int_{\gamma=0}^{2\pi} \rho(\alpha,\beta,\gamma) \sin\beta \, d\gamma \, d\beta \, d\alpha .$$

We will frequently be concerned with integrals of this form and adopt the abbreviation

$$\int_{\alpha=0}^{2\pi} \int_{\beta=0}^{\pi} \int_{\gamma=0}^{2\pi} f(\alpha,\beta,\gamma) \sin\beta \, d\gamma \, d\beta \, d\alpha = \int f(\alpha,\beta,\gamma) d\Omega .$$

When $f(\alpha,\beta,\gamma) = 1$ the integral becomes $\int d\Omega = 8\pi^2$.

Hence

$$\tilde{\rho} = \frac{1}{8\pi^2} \int \rho(\alpha,\beta,\gamma) d\Omega .$$

From Equation (36) this becomes

$$\tilde{\rho} = \frac{1}{8\pi^2} \int \sum_n \sum_m \sum_k c_n^m \alpha_n^k \overline{D_{km}^n(\alpha,\beta,\gamma)} d\Omega$$

$$= \frac{1}{8\pi^2} \sum_n \sum_m \sum_k c_n^m \alpha_n^k \int \overline{D_{km}^n(\alpha,\beta,\gamma)} d\Omega .$$

To evaluate the integral, use is made of Rose's Equation (4.60):

$$\int \overline{D_{km}^n(\alpha,\beta,\gamma)} \, D_{k'm'}^{n'}(\alpha,\beta,\gamma) d\Omega = \frac{8\pi^2}{2n+1} \delta_{n'}^{n} \delta_{m'}^{m} \delta_{k'}^{k} . \qquad (37)$$

Now, from Equation (35),

$$Y_0^0(\theta',\lambda') = \sum_{k=-0}^{+0} D_{k0}^0(\alpha,\beta,\gamma) Y_0^k(\theta,\lambda) = D_{00}^0(\alpha,\beta,\gamma) Y_0^0(\theta,\lambda) .$$

But $Y_0^0(\theta,\lambda) = \sqrt{1/4\pi}\ P_0^0(\cos\theta) = \sqrt{1/4\pi}$ and also $Y_0^0(\theta',\lambda') = \sqrt{1/4\pi}$, so that $D_{00}^0(\alpha,\beta,\gamma) = 1$. Hence, putting $n' = m' = k' = 0$ in (37),

$$\int \overline{D_{km}^n(\alpha,\beta,\gamma)} d\Omega = \frac{8\pi^2}{2n+1} \delta_0^n \delta_0^m \delta_0^k = 8\pi^2 \delta_0^n \delta_0^m \delta_0^k$$

and the expression for $\widetilde{\rho}$ becomes

$$\widetilde{\rho} = \frac{1}{8\pi^2} \sum_{n=2}^{\infty} \sum_m \sum_k c_n^m \alpha_n^k 8\pi^2 \delta_0^n \delta_0^m \delta_0^k = 0$$

because $n \geqslant 2$. Since $\widetilde{\rho} = 0$, its real and imaginary parts are also zero and $\widetilde{\delta D} = \widetilde{\delta C} = 0$. Hence, the mean values of the down-range and cross-range components of the impact error are both zero.

The remaining statistics to be computed are the variance $\widetilde{\delta D^2}$ of the down-range impact errors, the variance $\widetilde{\delta C^2}$ of the cross-range impact errors, and the covariance $\widetilde{\delta D \cdot \delta C}$. In terms of ρ and its conjugate $\bar{\rho}$, the down-range and cross-range components of impact error are

$$\delta D = +\frac{1}{2}(\rho + \bar{\rho})$$

$$\delta C = -\frac{i}{2}(\rho - \bar{\rho}) .$$

Hence

$$\delta D^2 = +\frac{1}{4}(\rho + \bar{\rho})(\rho + \bar{\rho})$$

$$\delta C^2 = -\frac{1}{4}(\rho - \bar{\rho})(\rho - \bar{\rho})$$

$$\delta D \cdot \delta C = -\frac{i}{4}(\rho + \bar{\rho})(\rho - \bar{\rho})$$

so that

$$\widetilde{\delta D^2} = +\frac{1}{32\pi^2}\int (\rho + \bar{\rho})(\rho + \bar{\rho})d\Omega$$

$$\widetilde{\delta C^2} = -\frac{1}{32\pi^2}\int (\rho - \bar{\rho})(\rho - \bar{\rho})d\Omega$$

$$\widetilde{\delta D \cdot \delta C} = -\frac{i}{32\pi^2}\int (\rho + \bar{\rho})(\rho - \bar{\rho})d\Omega \,.$$

The reduction of these expressions to the desired forms follows the general pattern used in reducing the expression for $\tilde{\rho}$. The computations are tedious but offer no special difficulties. The properties of the $D^n_{km}(\alpha,\beta,\gamma)$ that must be made use of are Equation (37) and

$$\int D^n_{km}(\alpha,\beta,\gamma)\, D^{n'}_{k'm'}(\alpha,\beta,\gamma)d\Omega = (-1)^{k'-m'} \frac{8\pi^2}{2n+1}\, \delta^n_{n'}\delta^m_{-m'}\delta^k_{-k'} \,.$$

This last expression is obtained by replacing m' and k' in Equation (37) with $-m'$ and $-k'$ and then applying Rose's Equation (4.22) to substitute

$$D^{n'}_{-k',-m'}(\alpha,\beta,\gamma) = (-1)^{k'-m'}\, \overline{D^{n'}_{k'm'}(\alpha,\beta,\gamma)} \,.$$

As a result of these operations it is found that

$$\widetilde{\delta D^2} = \sum_n a_n k_n$$

$$\widetilde{\delta C^2} = \sum_n b_n k_n \qquad (38)$$

$$\widetilde{\delta D \cdot \delta C} = 0 \,,$$

where the real positive quantities

$$k_n = \frac{R^2}{4\pi} \sum_{m=-n}^{+n} \alpha_n^m \overline{\alpha_n^m} \tag{39}$$

are the so-called "degree variances" of the geoidal height. The quantities a_n and b_n depend only on the nominal trajectory and are given in terms of the c_n^m of Equation (33) by

$$a_n = \frac{2\pi}{R^2} \frac{1}{2n+1} \sum_{m=-n}^{+n} c_n^m \left[c_n^m + (-1)^m c_n^{-m}\right]$$

$$b_n = \frac{2\pi}{R^2} \frac{1}{2n+1} \sum_{m=-n}^{+n} c_n^m \left[c_n^m - (-1)^m c_n^{-m}\right] .$$

Using the expression for the c_n^m derived earlier, these sums can be expressed as sums over non-negative values of m with the results

$$\begin{aligned} a_n &= g_3^2 + \frac{1}{2} n(n+1) g_1^2 + 2\mu R^{n-1} (g_3 I_n^0 + g_1 I_n^1) \\ &\quad + \mu^2 R^{2(n-1)} \left[(I_n^0)^2 + 2 \sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} (I_n^m)^2 \right] \\ b_n &= \frac{1}{2} n(n+1) g_2^2 + 2\mu R^{n-1} (g_2 J_n^1) \\ &\quad + \mu^2 R^{2(n-1)}\, 2 \sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} (J_n^m)^2 . \end{aligned} \tag{40}$$

To compute the a_n and b_n for a specific trajectory it is necessary first to obtain r and θ as functions of the time; then to integrate the adjoint Equation (11) to obtain the constants g_1, g_2, g_3 and the functions $f_1(t)$, $f_2(t)$, $f_3(t)$; and, finally, to evaluate numerically the integrals I_n^m and J_n^m in (30). The most difficult part of this task is to compute the associated Legendre functions $P_n^m(\cos\theta)$ for the values of $\theta = \theta(t)$ occurring along the trajectory. Some preliminary computations of these functions for large values of n (and m, since $0 \leqslant m \leqslant n$) suggested that the computations could be performed more accurately in terms of the normalized functions $\overline{P_n^m}(\cos\theta)$ (the bar here denotes normalization, not complex conjugation, the latter process not being involved in any of the subsequent operations) defined by

$$\overline{P_n^m} = \sqrt{\frac{2n+1}{2}\,\frac{(n-m)!}{(n+m)!}}\,P_n^m \,.$$

When these functions are introduced it is found that Equations (40) can be written in the forms

$$\begin{aligned}
a_n &= g_3^2 + \frac{1}{2}n(n+1)g_1^2 + 2\mu R^{n-1}\left(g_3\overline{I_n^0} + g_1\sqrt{n(n+1)}\,\overline{I_n^1}\right) \\
&\quad + \mu^2 R^{2(n-1)}\left[(\overline{I_n^0})^2 + 2\sum_{m=1}^{n}(\overline{I_n^m})^2\right] \\
b_n &= \frac{1}{2}n(n+1)g_2^2 + 2\mu R^{n-1}\left(g_2\sqrt{n(n+1)}\,\overline{J_n^1}\right) \\
&\quad + \mu^2 R^{2(n-1)}\left[2\sum_{m=1}^{n}(\overline{J_n^m})^2\right],
\end{aligned} \tag{41}$$

where the "normalized" integrals $\overline{I_n^m}$ and $\overline{J_n^m}$ are defined by $\overline{I_n^m} = \sqrt{(n-m)!/(n+m)!}\;I_n^m$ and $\overline{J_n^m} = \sqrt{(n-m)!/(n+m)!}\;J_n^m$ and may be written in the forms

$$\overline{I_n^m} = \sqrt{\frac{2}{2n+1}}\int_{t_0}^{T}\frac{1}{r^{n+2}}\left[f_1(t)\left(m\cot\theta\cdot\overline{P_n^m} - \sqrt{(n-m)(n+m+1)}\,\overline{P_n^{m+1}}\right) - f_3(t)(n+1)\overline{P_n^m}\right]dt$$

$$\overline{J_n^m} = \sqrt{\frac{2}{2n+1}}\int_{t_0}^{T}\frac{1}{r^{n+2}}f_2(t)\,m\csc\theta\cdot\overline{P_n^m}\,dt.$$

In computing the integrand functions the factors $\cot\theta$ and $\csc\theta$ would be expected to cause difficulty with the small values of θ occurring near the launch point, except for $m = 0$ when the terms involving these factors are absent. But for $m \neq 0$ the $\overline{P_n^m}(\cos\theta)$ carry $\sin\theta$ as a factor and the products $\cot\theta\cdot\overline{P_n^m}(\cos\theta)$ and $\csc\theta\cdot\overline{P_n^m}(\cos\theta)$ remain finite as θ approaches zero. To make use of this fact, another set of functions

$$\overline{Q_n^m}(\cos\theta) = \csc\theta\cdot\overline{P_n^m}(\cos\theta)$$

is defined (these functions should not be confused with the Legendre functions of the second kind). The integrals then become

$$\overline{I_n^m} = \sqrt{\frac{2}{2n+1}} \int_{t_0}^{T} \frac{1}{r^{n+2}} \left[f_1(t)\left(m \cos\theta \cdot \overline{Q_n^m} - \sqrt{(n-m)(n+m+1)}\,\overline{P_n^{m+1}}\right) - f_3(t)(n+1)\overline{P_n^m} \right] dt$$

$$\overline{J_n^m} = \sqrt{\frac{2}{2n+1}} \int_{t_0}^{T} \frac{1}{r^{n+2}} f_2(t) m \overline{Q_n^m}\, dt \, .$$

The values of the $\overline{P_n^m}$ are needed for $0 \leqslant m \leqslant n$ and the values of the $\overline{Q_n^m}$ are needed for $1 \leqslant m \leqslant n$. For each value of θ occurring along the trajectory, the values of the $\overline{P_n^m}$ and the $\overline{Q_n^m}$ were computed by the following recursion schemes:

For values of $0 \leqslant n \leqslant N$ and $0 \leqslant m \leqslant n$,

$$\overline{P_0^0} = \frac{1}{\sqrt{2}} \qquad \overline{P_1^0} = \sqrt{\frac{3}{2}} \cos\theta$$

$n = 2,3,\cdots,N$

$$\overline{P_n^0} = \frac{1}{n}\sqrt{(2n-1)(2n+1)} \cos\theta \cdot \overline{P_{n-1}^0} - \frac{n-1}{n}\sqrt{\frac{2n+1}{2n-3}}\,\overline{P_{n-2}^0}$$

$n = 1,2,\cdots,N$

$$\overline{P_n^n} = \sqrt{\frac{2n+1}{2n}} \sin\theta \cdot \overline{P_{n-1}^{n-1}}$$

$n = 2,3,\cdots,N$

$$\overline{P_n^{n-1}} = \frac{\sqrt{2n+1}}{n} \cos\theta \cdot \overline{P_{n-1}^{n-1}} + \frac{n-1}{n}\sqrt{\frac{2n+1}{2n-2}} \sin\theta \cdot \overline{P_{n-1}^{n-2}}$$

$m = 1 : n = 3,4.\cdots,N$

$m = 2 : n = 4,5,\cdots,N$

$\cdots$

$m = N-2 : n = N$

$$\overline{P_n^m} = \frac{2n-1}{n}\sqrt{\frac{(2n+1)(n-m)}{(2n-1)(n+m)}} \cos\theta \cdot \overline{P_{n-1}^m} - \frac{n-1}{n}\sqrt{\frac{(2n+1)(n-m)(n-m-1)}{(2n-3)(n+m)(n+m-1)}}\,\overline{P_{n-2}^m} + \frac{2n-1}{n}\, m \sqrt{\frac{2n+1}{(2n-1)(n+m)(n+m-1)}} \sin\theta \cdot \overline{P_{n-1}^{m-1}}$$

For values of $1 \leqslant n \leqslant N$ and $1 \leqslant m \leqslant n$,

$$\overline{Q_1^1} = \sqrt{\frac{3}{2}} \qquad \overline{Q_2^1} = \frac{\sqrt{15}}{2}\cos\theta$$

$n = 2,3,\cdots,N$

$$\overline{Q_n^1} = \frac{2n-1}{n-1}\sqrt{\frac{(2n+1)(n-1)}{(2n-1)(n+1)}}\cos\theta\cdot\overline{Q_{n-1}^1} - \frac{n}{n-1}\sqrt{\frac{(2n+1)(n-1)(n-2)}{(2n-3)(n+1)n}}\,\overline{Q_{n-2}^1}$$

$n = 2,3,\cdots,N$

$$\overline{Q_n^n} = \sqrt{\frac{2n+1}{2n}}\sin\theta\cdot\overline{Q_{n-1}^{n-1}}$$

$n = 3,4,\cdots,N$

$$\overline{Q_n^{n-1}} = \frac{\sqrt{2n+1}}{n}\cos\theta\cdot\overline{Q_{n-1}^{n-1}} + \frac{n-1}{n}\sqrt{\frac{2n+1}{2n-2}}\sin\theta\cdot\overline{Q_{n-1}^{n-2}}$$

$m = 2 : n = 4,5,\cdots,N$

$m = 3 : n = 5,6,\cdots,N$

$\cdots$

$m = N-2 : n = N$

$$\overline{Q_n^m} = \frac{2n-1}{n}\sqrt{\frac{(2n+1)(n-m)}{(2n-1)(n+m)}}\cos\theta\cdot\overline{Q_{n-1}^m} - \frac{n-1}{n}\sqrt{\frac{(2n+1)(n-m)(n-m-1)}{(2n-3)(n+m)(n+m-1)}}\,\overline{Q_{n-2}^m} + \frac{2n-1}{n}m\sqrt{\frac{2n+1}{(2n-1)(n+m)(n+m-1)}}\sin\theta\cdot\overline{Q_{n-1}^{m-1}}$$

X. TRAJECTORY COMPUTATIONS

For the trajectory computations it was assumed that the missile consisted of two powered stages, designated as Stages 1 and 2, and a reentry body designated as Stage 3. The gross characteristics of the three stages are listed in the following table

Stage	Initial Mass (kg)	Rate of Mass Decrease (kg/sec)	Thrust (newtons)	Cross-Section Area (m^2)
1	73754.0	795.	2035200.	6.132
2	9330.0	105.	286230.	2.775
3	72.1	0.	0.	0.117

The first stage was assumed to burn for a fixed time of 75 sec and the second stage for a variable additional time T_2 from 0 to 65 sec. To avoid considering details of the guidance system, it was assumed that the trajectories are zero lift in which the only aerodynamic force is the drag acting in a direction opposite to the velocity and the thrust (for the first two stages) acting in the same direction as the velocity. The vertical velocity at launch ($t_0 = 0$) was taken as 20 m/sec and the horizontal velocity at launch, V_H, was given a value in the range 0 - 4 m/sec. By varying the initial horizontal velocity and the burning time of the second stage, a family of trajectories having various ranges and total times of flight is obtained.

Drag data for the three stages were given initially in the form of tables of drag coefficient C_D vs. Mach number M. These values were fitted with Walton-Shanks formulas of the form

$$C_D = (1 + J)A(\lambda) + (1 - J)B(\lambda)$$

$$A(\lambda) = A_0 + A_1\lambda + A_2\lambda^2$$

$$B(\lambda) = B_0 + B_1\lambda$$

$$J = \frac{\lambda}{\sqrt{(1-\beta)\lambda^2 + \beta}}$$

$$\lambda = \frac{M^2 - \alpha}{M^2 + \alpha} .$$

The constants have the following values.

Stage	1	2	3
α	+8.8212-01	+1.0808-00	+8.1741-01
β	+1.0618-02	+2.5982-02	+2.2353-05
A_0	+1.7087-01	+2.0935-01	+2.3078-01
A_1	-3.2897-02	-9.6134-02	-3.0271-02
A_2	-1.4217-02	+2.1966-02	-1.3220-01
B_0	+2.7938-02	+1.0241-02	+1.5429-01
B_1	-2.0284-03	-1.2650-02	+1.7857-02

The derivative $C'_D(M)$, required in integrating the adjoint equation, is obtained by differentiating the Walton-Shanks formula.

The atmospheric density and speed of sound as functions of geopotential altitude were taken from the tables of the U. S. Standard Atmosphere with the following modifications. For geometric altitudes greater than 90,000 meters, the Standard Atmosphere is defined in a different and more complicated manner than it is for lower altitudes. To allow a uniform mode of computation to be used at all altitudes, this portion of the Standard Atmosphere was altered slightly. However, the density at these altitudes is so small that the change has a negligible effect on the results. Another change made in the Standard Atmosphere was to replace the polygonal function defining the molecular scale temperature T_M as a function of geopotential altitude by an eighth degree polynomial giving $1/T_M$ as a function of geopotential altitude. With these changes, the derivatives $\rho'(h)$ and $a'(h)$ required in integrating the adjoint equation could be obtained analytically. The atmospheric density was taken as zero at geometric altitudes $\geqslant$ 150000 meters.

The trajectory computations were made with a fourth order Runge-Kutta integration scheme. The time increments were taken as follows: 0.25 sec from $t = 0$ to $t = 5$ sec, 1.25 sec from 5 sec to 75 sec, 5 sec from 75 sec to an altitude of approximately 150,000 meters, and 0.25 sec from that point until

impact. One exception to this scheme occurs at the termination of second stage thrust, where two smaller intervals whose sum is 5 sec were used to allow thrust termination to occur at an arbitrarily chosen time.

Using a set of arbitrarily chosen "even" values of T_2 and V_H, a family of trajectories was computed having the ranges and times of flight plotted in Figure 5. Using numerical values of partial derivatives obtained from these trajectories, values of T_2 and V_H were derived by an iteration process to produce the final 45 trajectories listed in the following table. In this table T denotes the total time-of-flight in seconds and R the range in nautical miles.

T	R	T_2	V_H
1280	0	28.68013141	0.00000000
	500	29.75426850	0.55712446
	1000	32.78836295	1.11086563
	1500	37.29593791	1.65948124
	2000	42.67621520	2.20437465
	2500	48.40687876	2.75286094
	3000	54.23841118	3.32571427
1600	0	43.16535414	0.00000000
	500	43.58334483	0.41185108
	1000	44.79389359	0.82380098
	1500	46.67771033	1.23624173
	2000	49.06795304	1.65010040
	2500	51.78537344	2.06705416
	3000	54.67051088	2.48996986
	3500	57.61244696	2.92502477
	4000	60.59935459	3.38334173
1920	0	52.21427363	0.00000000
	500	52.40753281	0.33187979
	1000	52.97353510	0.66433233
	1500	53.87333100	0.99799642
	2000	55.04924451	1.33362982
	2500	56.43370819	1.67215207
	3000	57.95835115	2.01467201
	3500	59.56213362	2.36260283
	4000	61.19814676	2.71812983
	4500	62.84295251	3.08608459
	5000	64.52198318	3.47026982

RANGE AND TIME OF FLIGHT VS T_2 AND V_H

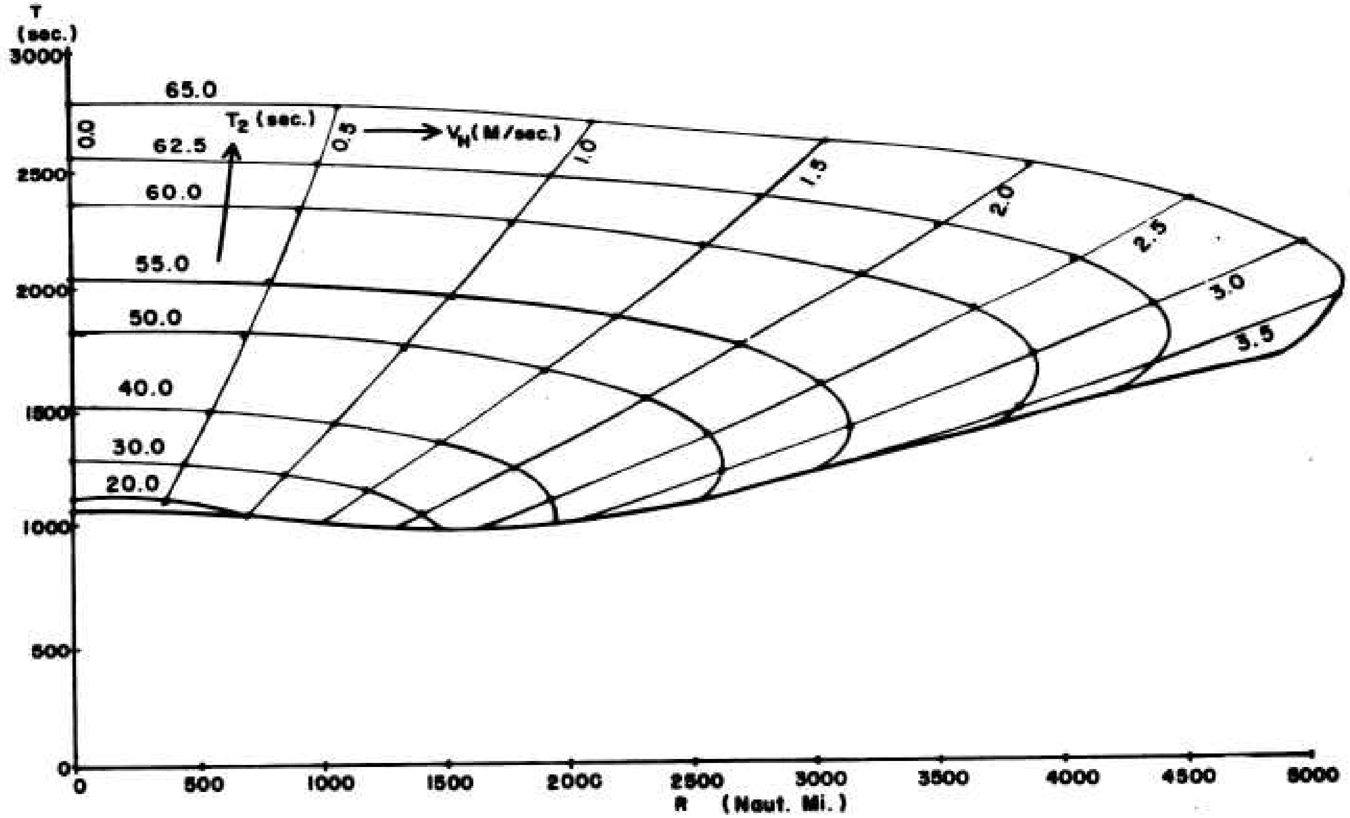

FIGURE 5

T	R	T_2	V_H
2240	0	58.22928795	0.00000000
	500	58.33024376	0.28248605
	1000	58.62760481	0.56551324
	1500	59.10555530	0.84963652
	2000	59.73994425	1.13543461
	2500	60.50114277	1.42351394
	3000	61.35721867	1.71450737
	3500	62.27702102	2.00907542
	4000	63.23288008	2.30793586
	4500	64.20281645	2.61201070
2560	0	62.44253642	0.00000000
	500	62.50011875	0.24938435
	1000	62.67025943	0.49920811
	1500	62.94540583	0.74991005
	2000	63.31381211	1.00192618
	2500	63.76065880	1.25568501
	3000	64.26937279	1.51159996
	3500	64.82298548	1.77006024

For all of these trajectories, the thrust is terminated at an altitude greater than 150,000 meters, justifying the assumption that the non-gravitational acceleration is zero from the time of thrust termination to reentry. For each trajectory the position and velocity components at the time of impact were used in formulas (12) to obtain the values of a and b which were required later to compute the constants g_1, g_2, g_3 of Equations (27) and the functions $f_1(t)$, $f_2(t)$, $f_3(t)$ defined by Equations (29).

To allow the integration of the adjoint equation it would, in principle, be necessary to retain from the integration of the nominal trajectory only the values of r(t) and $\theta(t)$ for $t_0 \leq t \leq T$, but it is more efficient to retain the values of r, $\sin\theta$, $\cos\theta$ (θ itself is not actually required), $\sin\varphi$, $\cos\varphi$, D, m, and v. In addition, although not actually required during the integration of the nominal trajectory, the values of P and Q are computed from Equations (8) and retained for use during the integration of the adjoint equation.

The adjoint equation is solved by a Runge-Kutta integration (backwards in time) using an integration step of 0.5 sec from t = T to a time near that of reentry and a step size of 10 sec from there to the time of launch. This integration gives $Y(t,T)$ for $T \geq t \geq t_0$ and hence $X(T,t) = Y^T(t,T)$ for $t_0 \leq t \leq T$. As a check on the solution of the adjoint equation, it was solved for the trajectory T = 1920 sec and R = 3000 nautical miles and the final matrix $X(T,t_0)$ printed

out. Then the nominal trajectory program was rerun and the values of the non-gravitational acceleration components stored in the permanent file. Trajectories were then run with varied initial conditions where, instead of computing the non-gravitational acceleration components, they were read from the permanent file for the times prior to reentry and computed anew from reentry to the time of nominal impact. From the differences between the position and velocity components at the nominal time of impact for the varied trajectories and the nominal trajectory, it was possible to compute numerical values of 16 of the 20 non-zero elements of $X(T, t_0)$. The remaining four non-zero elements could not be obtained by this method because the trajectory program did not allow out-of-plane variations of the initial conditions. The values of the 16 non-zero elements of $X(T,t_0)$ agreed with those obtained from the solution of the adjoint equation to four or more significant figures.

When this method of checking was first attempted, the polygonal T_M function of the Standard Atmosphere was in use and the two methods gave results agreeing to only one to three significant figures. It was for this reason that the polynomial approximation to $1/T_M$ was adopted, since this eliminated the discontinuities in the derivatives occurring with the polygonal function. A similar difficulty might have been experienced by employing the tables of drag coefficient originally available, but the use of the Walton-Shanks representation of the drag coefficient at the outset avoided the problem.

In the integration of the adjoint equation, 36 elements of the $X(t,t_0)$ matrix are determined (16 of which are zero), but not all of these are required to compute the a_n and b_n. The elements actually required are those appearing in the functions $f_1(t)$, $f_2(t)$, $f_3(t)$ defined by Equations (29) and those (for $t = t_0$ only) appearing in the constants of Equations (27). These values were stored in the permanent file for later use in the computation of the a_n and b_n. The values of r, $\sin\theta$, $\cos\theta$ which had been developed during the integration of the nominal trajectory and used during the integration of the adjoint equation were also put in the permanent file along with $f_1(t)$, $f_2(t)$, $f_3(t)$.

The major part of the computation of the a_n and b_n consists of the evaluation of the integrals $\overline{I_n^m}$ and $\overline{J_n^m}$. This was done by Simpson's Rule, using a 10-sec interval. In this stage of the computation it was convenient to replace r, the radius vector of the missile, by the normalized value r/R, with R the radius of the reference sphere; to take R = 1 in the formulas (41) for a_n and b_n; and to express μ in these equations in units of (reference sphere radius)^3/sec^2. The values of $f_1(t)$,$f_2(t)$,$f_3(t)$, r, $\sin\theta$, $\cos\theta$ required to evaluate the integrals were read from the permanent file created during the integration of the adjoint equation. A subroutine

of the program for computing the a_n and b_n employed the recursion formulas which have been presented earlier for the computation of the $\overline{P_n^m}$ and the $\overline{Q_n^m}$.

To allow a break-down of the total errors into navigation and in-flight errors, a_n and b_n in Equations (41) are written in the forms

$$a_n = a_{1n} + a_{2n} + a_{3n} + a_{4n}$$

where

$$\begin{aligned}
a_{1n} &= g_3^2 \\
a_{2n} &= \frac{1}{2} n(n+1) g_1^2 \\
a_{3n} &= 2\mu R^{n-1}\left(g_3 \overline{I_n^0} + g_1 \sqrt{n(n+1)}\, \overline{I_n^1}\right) \\
a_{4n} &= \mu^2 R^{2(n-1)} \left[\left(\overline{I_n^0}\right)^2 + 2 \sum_{m=1}^{n} \left(\overline{I_n^m}\right)^2\right]
\end{aligned}$$

$$b_n = b_{2n} + b_{3n} + b_{4n}$$

where

$$\begin{aligned}
b_{2n} &= \frac{1}{2} n(n+1) g_2^2 \\
b_{3n} &= 2\mu R^{n-1}\left(g_2 \sqrt{n(n+1)}\, \overline{J_n^1}\right) \\
b_{4n} &= \mu^2 R^{2(n-1)} \left[2 \sum_{m=1}^{n} \left(\overline{J_n^m}\right)^2\right].
\end{aligned}$$

An examination of the origins of these terms shows that if the effect of in-flight errors only is to be computed, then a_n should be taken equal to a_{4n} and b_n equal to b_{4n}. If the effect of navigation errors only is to be computed, then a_n should be taken equal to $a_{1n} + a_{2n}$ and b_n equal to b_{2n}. The effect of navigation errors only can be further broken down into the effect of horizontal errors only ($a_n = a_{2n}$ and $b_n = b_{2n}$) and the effect of height errors only ($a_n = a_{1n}$ and $b_n = 0$). It will be noted that a_{1n}, a_{2n}, a_{4n}, b_{2n}, b_{4n} are necessarily positive whereas a_{3n} and b_{3n} are not. These last two values represent the effects on the total mean square impact errors $\widetilde{\delta D^2}$ and $\widetilde{\delta C^2}$ of correlations between the navigation

and in-flight effects. If a_{3n} and b_{3n} are zero, the total effect is the statistical sum of the separate navigation and in-flight effects. If a_{3n} or b_{3n} is positive, the navigation and in-flight effects (for the corresponding component of impact error) tend to reinforce one another and if negative tend to cancel one another.

To allow all of these effects to be evaluated separately, if desired, the program for computing the a_n and b_n for each trajectory stores on magnetic tape the value of a_{1n} (actually independent of n) and for each value of n, $2 \leq n \leq 174$, the values of a_{2n}, a_{3n}, a_{4n}, b_{2n}, b_{3n}, b_{4n}. All of these quantities are dimensionless, so that when they are used in Equations (38) the resulting values of $\widehat{\delta D^2}$ and $\widehat{\delta C^2}$ are expressed in the same units as the degree variances k_n. The maximum value of $n = 174$ was imposed by the core storage available in the CDC 6700 Computer. The results therefore reflect the influence of geoidal height wavelengths greater than about 120 nautical miles. The value of n could be increased, and the influence of shorter wavelengths represented, by modifying the a_n and b_n program to utilize the disk storage. This in itself would be expected to increase the already long running time of the program and it would be increased to an even greater extent as a result of the larger value of n, since the number of values of the $\overline{P}_n^m$ and $\overline{Q}_n^m$ that must be computed is approximately proportional to n^2.

XI. NUMERICAL RESULTS

As an example of the computation of impact errors from Equations (38), the degree variances k_n (in meters^2) have been computed from the empirical formula

$$k_n = a \frac{e^{-b(n-2)}}{n^c (n-1)^2} \tag{43}$$

with the numerical values a = 6510, b = 0.00117, c = 1.058. This formula and the values of the parameters have been obtained by E. D. Ball as preliminary results of a study now in progress on the combination of satellite and surface gravity data.

The following table is a listing of the results obtained for the trajectory having a range R = 3000 nautical miles and a time of flight T = 1920 sec. Similar listings have been produced for all 45 trajectories used in the study. To interpret the listing, consider the last three entries (NAV AND IN-FLIGHT) in the last line (n = 2). The value RMSD = $\sqrt{\overline{\delta D^2}}$ = 122.31 meters is the root mean square down-range impact error that would result if the actual field of the earth had degree variances given by formula (43) (for $2 \leqslant n \leqslant 174$, the higher degree variances being zero) but only the normal field were taken into account by navigation and guidance. The value RMSC = $\sqrt{\overline{\delta C^2}}$ = 89.90 is the root mean square cross-range impact error under the same conditions. The value RMSR = $\sqrt{\overline{\delta D^2} + \overline{\delta C^2}}$ = 151.80 is the root mean square radial impact error. All values of RMSR in meters can be converted approximately into circular probable errors in yards by the expression CPE(yd) = 0.91 RMSR(m).

Under IN-FLIGHT ONLY the entries RMSD = 244.66, RMSC = 126.30, RMSR = 275.34 are the errors that would result if the actual field were the same as before and if navigation accounted for this field completely but guidance used only the normal field. Under ALL NAV ONLY, the entries RMSD = 145.64, RMSC = 49.20, RMSR = 153.72 are the errors that would result if the actual field were again the same and if guidance accounted for this field completely but navigation used only the normal field.

The navigation only errors are further broken down in the remaining entries. Those headed NAV HEIGHT ONLY are the impact errors which would result if navigation took into account the vertical deflections of the actual field but failed to account for the geoidal heights. Those headed NAV HORIZ ONLY are the impact errors which would result if navigation took the geoidal heights of the actual field into account but neglected the vertical deflections.

TIME OF FLIGHT 1320 SECONDS   RANGE 3000 NAUTICAL MILES   ERRORS IN METERS

	NAV HEIGHT ONLY			NAV HORIZ ONLY			ALL NAV ONLY			IN-FLIGHT ONLY			NAV AND IN-FLIGHT		
	RMS D	RMS C	RMS R	RMS D	RMS C	RMS R	RMS D	RMS C	RMS R	RMS D	RMS C	RMS R	RMS D	RMS C	RMS R
174	.06	0.00	.06	.22	1.04	1.07	.23	1.04	1.07	.65	.28	.71	.46	.76	.90
173	.09	0.00	.09	.31	1.48	1.51	.32	1.48	1.51	.93	.40	1.01	.68	1.06	1.28
172	.11	0.00	.11	.38	1.81	1.85	.40	1.81	1.85	1.14	.49	1.24	.83	1.33	1.57
171	.13	0.00	.13	.44	2.10	2.14	.46	2.10	2.15	1.32	.57	1.44	.97	1.53	1.81
170	.14	0.00	.14	.49	2.35	2.40	.51	2.35	2.40	1.46	.64	1.62	1.08	1.72	2.03
169	.16	0.00	.15	.54	2.58	2.63	.56	2.58	2.64	1.63	.70	1.77	1.19	1.88	2.23
168	.17	0.00	.17	.58	2.79	2.85	.61	2.79	2.85	1.77	.76	1.92	1.29	2.04	2.41
167	.19	0.00	.19	.65	2.99	3.05	.65	2.99	3.06	1.69	.82	2.06	1.38	2.18	2.58
166	.20	0.00	.20	.67	3.17	3.24	.69	3.17	3.25	2.02	.87	2.19	1.47	2.32	2.74
165	.21	0.00	.21	.70	3.35	3.43	.73	3.35	3.43	2.13	.92	2.32	1.56	2.45	2.90
164	.22	0.00	.22	.74	3.52	3.60	.77	3.52	3.61	2.24	.97	2.44	1.64	2.57	3.05
163	.23	0.00	.23	.77	3.69	3.77	.81	3.69	3.77	2.35	1.01	2.56	1.72	2.69	3.19
162	.24	0.00	.24	.81	3.84	3.93	.84	3.84	3.94	2.45	1.06	2.67	1.79	2.80	3.32
161	.25	0.00	.25	.84	4.00	4.08	.88	4.00	4.09	2.55	1.10	2.78	1.87	2.91	3.46
160	.26	0.00	.26	.87	4.15	4.24	.91	4.15	4.24	2.65	1.14	2.89	1.94	3.02	3.59
159	.27	0.00	.27	.90	4.29	4.38	.94	4.29	4.39	2.75	1.18	2.99	2.01	3.12	3.71
158	.28	0.00	.28	.93	4.43	4.53	.97	4.43	4.54	2.84	1.22	3.09	2.08	3.22	3.83
157	.29	0.00	.29	.96	4.57	4.67	1.00	4.57	4.68	2.93	1.26	3.20	2.14	3.32	3.95
156	.30	0.00	.30	.99	4.70	4.81	1.03	4.70	4.82	3.02	1.30	3.29	2.21	3.42	4.07
155	.31	0.00	.31	1.01	4.84	4.94	1.06	4.84	4.95	3.11	1.34	3.39	2.28	3.51	4.18
154	.32	0.00	.32	1.04	4.97	5.07	1.09	4.97	5.08	3.20	1.39	3.49	2.34	3.60	4.29
153	.33	0.00	.33	1.07	5.09	5.20	1.12	5.09	5.21	3.29	1.42	3.58	2.40	3.69	4.41
152	.34	0.00	.34	1.09	5.22	5.33	1.15	5.22	5.34	3.37	1.45	3.67	2.47	3.78	4.51
151	.35	0.00	.35	1.12	5.34	5.45	1.17	5.34	5.47	3.46	1.49	3.77	2.53	3.87	4.62
150	.36	0.00	.36	1.14	5.46	5.58	1.20	5.46	5.59	3.54	1.53	3.86	2.59	3.95	4.73
149	.37	0.00	.37	1.17	5.58	5.70	1.23	5.58	5.71	3.62	1.56	3.95	2.65	4.04	4.83
148	.38	0.00	.38	1.19	5.70	5.82	1.25	5.70	5.84	3.71	1.60	4.04	2.71	4.12	4.93
147	.39	0.00	.39	1.22	5.82	5.94	1.28	5.82	5.96	3.79	1.63	4.12	2.77	4.20	5.03
146	.39	0.00	.39	1.24	5.93	6.06	1.30	5.93	6.07	3.87	1.67	4.21	2.83	4.28	5.13
145	.40	0.00	.40	1.27	6.05	6.18	1.33	6.05	6.19	3.95	1.70	4.30	2.89	4.36	5.23
144	.41	0.00	.41	1.29	6.16	6.29	1.35	6.16	6.31	4.03	1.74	4.39	2.95	4.44	5.33
143	.42	0.00	.42	1.31	6.27	6.41	1.38	6.27	6.42	4.11	1.77	4.47	3.00	4.52	5.43
142	.43	0.00	.43	1.34	6.38	6.52	1.41	6.38	6.54	4.19	1.81	4.56	3.06	4.60	5.52
141	.44	0.00	.44	1.36	6.49	6.63	1.43	6.49	6.65	4.26	1.84	4.64	3.12	4.67	5.62
140	.45	0.00	.45	1.38	6.60	6.74	1.45	6.60	6.76	4.34	1.87	4.73	3.18	4.75	5.71
139	.46	0.00	.46	1.41	6.71	6.86	1.48	6.71	6.87	4.42	1.91	4.81	3.23	4.83	5.81
138	.47	0.00	.47	1.43	6.82	6.97	1.50	6.82	6.98	4.50	1.94	4.90	3.29	4.90	5.90

137	.45	0.00	.45	1.45	6.92	7.07	1.53	6.92	7.09	4.55	1.98	4.99	3.35	4.37	6.00
136	.49	0.00	.49	1.47	7.03	7.18	1.55	7.03	7.20	4.65	2.01	5.07	3.40	5.05	6.09
135	.50	0.00	.50	1.50	7.14	7.29	1.58	7.14	7.31	4.73	2.04	5.15	3.46	5.12	6.18
134	.50	0.00	.50	1.52	7.24	7.40	1.60	7.24	7.42	4.81	2.08	5.24	3.52	5.19	6.27
133	.51	0.00	.51	1.54	7.35	7.50	1.62	7.35	7.52	4.89	2.11	5.32	3.57	5.26	6.36
132	.52	0.00	.52	1.56	7.45	7.61	1.65	7.45	7.63	4.95	2.14	5.41	3.63	5.33	6.45
131	.53	0.00	.53	1.58	7.55	7.72	1.67	7.55	7.74	5.04	2.18	5.49	3.69	5.40	6.54
130	.54	0.00	.54	1.60	7.66	7.82	1.69	7.65	7.84	5.12	2.21	5.57	3.74	5.47	6.63
129	.55	0.00	.55	1.63	7.76	7.93	1.72	7.76	7.95	5.19	2.24	5.66	3.80	5.54	6.72
128	.56	0.00	.56	1.65	7.86	8.03	1.74	7.86	8.05	5.27	2.28	5.74	3.86	5.61	6.81
127	.57	0.00	.57	1.67	7.96	8.14	1.76	7.96	8.16	5.35	2.31	5.83	3.91	5.68	6.90
126	.58	0.00	.58	1.69	8.06	8.24	1.79	8.06	8.26	5.43	2.34	5.91	3.97	5.75	6.99
125	.59	0.00	.59	1.71	8.17	8.34	1.81	8.17	8.36	5.50	2.38	5.99	4.03	5.82	7.08
124	.60	0.00	.60	1.73	8.27	8.45	1.83	8.27	8.47	5.58	2.41	6.08	4.09	5.89	7.17
123	.61	0.00	.61	1.75	8.37	8.55	1.86	8.37	8.57	5.66	2.45	6.16	4.14	5.95	7.25
122	.62	0.00	.62	1.77	8.47	8.65	1.88	8.47	8.67	5.74	2.48	6.25	4.20	6.02	7.34
121	.63	0.00	.63	1.80	8.57	8.75	1.90	8.57	8.78	5.82	2.51	6.34	4.26	6.09	7.43
120	.64	0.00	.64	1.82	8.67	8.86	1.93	8.67	8.88	5.89	2.55	6.42	4.32	6.16	7.52
119	.65	0.00	.65	1.84	8.77	8.96	1.95	8.77	8.98	5.97	2.58	6.51	4.37	6.22	7.60
118	.66	0.00	.66	1.86	8.87	9.06	1.97	8.87	9.09	6.05	2.62	6.59	4.43	6.29	7.69
117	.67	0.00	.67	1.88	8.97	9.16	2.00	8.97	9.19	6.13	2.65	6.68	4.49	6.35	7.78
116	.69	0.00	.69	1.90	9.07	9.27	2.02	9.07	9.29	6.21	2.69	6.77	4.55	6.42	7.87
115	.70	0.00	.70	1.92	9.17	9.37	2.04	9.17	9.39	6.29	2.72	6.86	4.61	6.49	7.96
114	.71	0.00	.71	1.94	9.27	9.47	2.07	9.27	9.50	6.37	2.75	6.94	4.67	6.55	8.04
113	.72	0.00	.72	1.96	9.37	9.57	2.09	9.37	9.60	6.45	2.79	7.03	4.73	6.62	8.13
112	.73	0.00	.73	1.98	9.47	9.67	2.11	9.47	9.70	6.54	2.83	7.12	4.79	6.68	8.22
111	.74	0.00	.74	2.01	9.57	9.78	2.14	9.57	9.80	6.62	2.86	7.21	4.85	6.75	8.31
110	.75	0.00	.75	2.03	9.67	9.88	2.16	9.67	9.91	6.70	2.90	7.30	4.91	6.81	8.39
109	.77	0.00	.77	2.05	9.77	9.98	2.19	9.77	10.01	6.78	2.94	7.39	4.97	6.88	8.48
108	.78	0.00	.78	2.07	9.87	10.08	2.21	9.87	10.11	6.87	2.97	7.48	5.03	6.94	8.57
107	.79	0.00	.79	2.09	9.97	10.18	2.23	9.97	10.22	6.95	3.01	7.57	5.09	7.00	8.66
106	.80	0.00	.80	2.11	10.07	10.29	2.26	10.07	10.32	7.03	3.05	7.67	5.15	7.07	8.75
105	.82	0.00	.82	2.13	10.17	10.39	2.28	10.17	10.42	7.12	3.08	7.75	5.21	7.13	8.84
104	.83	0.00	.83	2.15	10.27	10.49	2.31	10.27	10.53	7.21	3.12	7.85	5.28	7.20	8.92
103	.84	0.00	.84	2.17	10.37	10.60	2.33	10.37	10.63	7.29	3.16	7.94	5.34	7.26	9.01
102	.85	0.00	.85	2.19	10.47	10.70	2.36	10.47	10.73	7.38	3.20	8.04	5.40	7.33	9.10
101	.87	0.00	.87	2.22	10.57	10.80	2.38	10.57	10.84	7.47	3.23	8.14	5.47	7.39	9.19
100	.88	0.00	.88	2.24	10.67	10.91	2.40	10.67	10.94	7.56	3.27	8.23	5.53	7.45	9.28
99	.90	0.00	.90	2.26	10.78	11.01	2.43	10.78	11.05	7.64	3.31	8.33	5.60	7.52	9.37
98	.91	0.00	.91	2.28	10.88	11.11	2.45	10.88	11.15	7.73	3.35	8.43	5.66	7.58	9.46
97	.92	0.00	.92	2.30	10.98	11.22	2.48	10.98	11.26	7.83	3.39	8.53	5.73	7.64	9.55
96	.94	0.00	.94	2.32	11.08	11.32	2.51	11.08	11.36	7.92	3.43	8.63	5.80	7.71	9.65
95	.95	0.00	.95	2.34	11.19	11.43	2.53	11.19	11.47	8.01	3.47	8.73	5.87	7.77	9.74
94	.97	0.00	.97	2.37	11.29	11.53	2.56	11.29	11.58	8.11	3.51	8.83	5.93	7.84	9.83
93	.99	0.00	.99	2.39	11.39	11.64	2.58	11.39	11.68	8.20	3.55	8.93	6.00	7.90	9.92
92	1.00	0.00	1.00	2.41	11.50	11.75	2.61	11.50	11.79	8.29	3.60	9.04	6.07	7.96	10.01

91	1.02	0.00	1.02	2.43	11.60	11.85	2.64	11.60	11.90	8.39	3.64	9.14	6.14	8.03	10.11
90	1.03	0.00	1.03	2.45	11.71	11.96	2.66	11.71	12.01	8.49	3.68	9.25	6.21	8.09	10.20
89	1.05	0.00	1.05	2.48	11.81	12.07	2.69	11.81	12.11	8.59	3.72	9.36	6.29	8.15	10.30
88	1.07	0.00	1.07	2.50	11.92	12.18	2.72	11.92	12.22	8.69	3.77	9.47	6.36	8.22	10.39
87	1.08	0.00	1.08	2.52	12.02	12.29	2.74	12.02	12.33	8.79	3.81	9.59	6.43	8.28	10.49
86	1.10	0.00	1.10	2.54	12.13	12.39	2.77	12.13	12.44	8.89	3.86	9.69	6.50	8.35	10.58
85	1.12	0.00	1.12	2.57	12.24	12.50	2.80	12.24	12.55	8.99	3.90	9.80	6.58	8.41	10.68
84	1.14	0.00	1.14	2.59	12.35	12.62	2.83	12.35	12.67	9.09	3.95	9.91	6.66	8.47	10.78
83	1.16	0.00	1.16	2.61	12.46	12.73	2.86	12.46	12.78	9.20	3.99	10.03	6.73	8.54	10.87
82	1.18	0.00	1.18	2.63	12.56	12.84	2.88	12.56	12.89	9.30	4.04	10.14	6.81	8.60	10.97
81	1.20	0.00	1.20	2.66	12.67	12.95	2.91	12.67	13.01	9.41	4.09	10.26	6.89	8.67	11.07
80	1.22	0.00	1.22	2.68	12.79	13.06	2.94	12.79	13.12	9.52	4.14	10.38	6.97	8.73	11.17
79	1.24	0.00	1.24	2.70	12.90	13.18	2.97	12.90	13.24	9.63	4.19	10.50	7.05	8.80	11.27
78	1.26	0.00	1.26	2.73	13.01	13.29	3.00	13.01	13.35	9.74	4.24	10.63	7.13	8.86	11.37
77	1.28	0.00	1.28	2.75	13.12	13.41	3.03	13.12	13.47	9.86	4.29	10.75	7.21	8.92	11.48
76	1.30	0.00	1.30	2.77	13.24	13.52	3.06	13.24	13.59	9.97	4.34	10.88	7.30	8.99	11.58
75	1.32	0.00	1.32	2.80	13.35	13.64	3.10	13.35	13.70	10.09	4.39	11.01	7.38	9.05	11.68
74	1.35	0.00	1.35	2.82	13.46	13.76	3.13	13.46	13.82	10.21	4.44	11.14	7.47	9.12	11.79
73	1.37	0.00	1.37	2.85	13.58	13.88	3.16	13.58	13.94	10.33	4.50	11.27	7.56	9.18	11.89
72	1.40	0.00	1.40	2.87	13.70	14.00	3.19	13.70	14.07	10.45	4.55	11.40	7.65	9.25	12.00
71	1.42	0.00	1.42	2.90	13.82	14.12	3.23	13.82	14.19	10.58	4.61	11.54	7.74	9.32	12.11
70	1.45	0.00	1.45	2.92	13.93	14.24	3.26	13.93	14.31	10.71	4.66	11.68	7.83	9.38	12.22
69	1.47	0.00	1.47	2.95	14.05	14.36	3.29	14.05	14.44	10.84	4.72	11.82	7.92	9.45	12.33
68	1.50	0.00	1.50	2.97	14.18	14.48	3.33	14.18	14.56	10.97	4.78	11.96	8.02	9.51	12.44
67	1.53	0.00	1.53	3.00	14.30	14.61	3.36	14.30	14.69	11.10	4.84	12.11	8.11	9.58	12.55
66	1.56	0.00	1.56	3.02	14.42	14.73	3.40	14.42	14.82	11.24	4.90	12.26	8.21	9.64	12.67
65	1.59	0.00	1.59	3.05	14.55	14.86	3.44	14.55	14.95	11.37	4.96	12.41	8.31	9.71	12.78
64	1.62	0.00	1.62	3.08	14.67	14.99	3.48	14.67	15.08	11.52	5.02	12.56	8.41	9.78	12.90
63	1.65	0.00	1.65	3.10	14.80	15.12	3.51	14.80	15.21	11.66	5.09	12.72	8.51	9.84	13.02
62	1.68	0.00	1.68	3.13	14.93	15.25	3.55	14.93	15.34	11.81	5.15	12.88	8.62	9.91	13.13
61	1.72	0.00	1.72	3.16	15.06	15.38	3.59	15.06	15.48	11.95	5.22	13.04	8.73	9.98	13.26
60	1.75	0.00	1.75	3.18	15.19	15.52	3.63	15.19	15.61	12.11	5.29	13.21	8.83	10.05	13.38
59	1.79	0.00	1.79	3.21	15.32	15.65	3.68	15.32	15.75	12.26	5.36	13.38	8.95	10.11	13.50
58	1.83	0.00	1.83	3.24	15.45	15.79	3.72	15.45	15.89	12.42	5.43	13.56	9.06	10.18	13.63
57	1.87	0.00	1.87	3.27	15.59	15.93	3.76	15.59	16.04	12.58	5.50	13.73	9.17	10.25	13.76
56	1.91	0.00	1.91	3.30	15.73	16.07	3.81	15.73	16.18	12.75	5.57	13.91	9.29	10.32	13.89
55	1.95	0.00	1.95	3.33	15.86	16.21	3.85	15.86	16.33	12.92	5.65	14.10	9.41	10.39	14.02
54	1.99	0.00	1.99	3.35	16.00	16.35	3.90	16.00	16.47	13.09	5.73	14.29	9.54	10.45	14.15
53	2.03	0.00	2.03	3.38	16.15	16.50	3.95	16.15	16.62	13.27	5.81	14.49	9.66	10.52	14.29
52	2.08	0.00	2.08	3.41	16.29	16.65	4.00	16.29	16.78	13.45	5.89	14.69	9.79	10.59	14.43
51	2.13	0.00	2.13	3.45	16.44	16.80	4.05	16.44	16.93	13.64	5.97	14.89	9.92	10.66	14.57
50	2.18	0.00	2.18	3.48	16.59	16.95	4.10	16.59	17.09	13.83	6.06	15.10	10.06	10.73	14.71
49	2.23	0.00	2.23	3.51	16.74	17.10	4.16	16.74	17.25	14.03	6.15	15.32	10.20	10.80	14.85
48	2.29	0.00	2.29	3.54	16.89	17.26	4.21	16.89	17.41	14.23	6.24	15.54	10.34	10.87	15.00
47	2.34	0.00	2.34	3.57	17.05	17.42	4.27	17.05	17.57	14.44	6.33	15.77	10.48	10.94	15.15
46	2.40	0.00	2.40	3.61	17.20	17.58	4.33	17.20	17.74	14.65	6.43	16.00	10.63	11.01	15.31

45	2.46	0.00	2.46	3.54	17.36	17.74	4.39	17.36	17.91	14.8[illegible]	6.53	15.24	10.79	11.08	15.47
44	2.53	0.00	2.53	3.57	17.53	17.91	4.45	17.53	18.19	15.10	6.65	16.49	10.95	11.15	15.63
43	2.60	0.00	2.60	3.71	17.59	18.08	4.53	17.69	18.25	15.37	6.74	16.75	11.11	11.22	15.79
42	2.67	0.00	2.67	3.74	17.82	18.22	4.60	17.86	18.45	15.57	6.85	17.01	11.27	11.30	15.96
41	2.74	0.00	2.74	3.76	18.04	18.43	4.67	18.04	18.63	15.82	6.96	17.23	11.45	11.37	16.13
40	2.82	0.00	2.82	3.82	18.21	18.61	[illegible]	18.21	18.82	16.09	7.08	17.57	11.62	11.44	16.31
39	2.90	0.00	2.90	3.85	18.39	18.79	4.83	18.39	19.01	16.35	7.20	17.86	11.81	11.51	16.49
38	2.99	0.00	2.99	3.89	18.56	18.98	4.91	18.56	19.21	16.67	7.33	18.17	12.00	11.58	16.68
37	3.06	0.00	3.06	3.93	18.76	19.17	5.00	18.76	19.42	16.91	7.45	18.45	12.19	11.66	16.87
36	3.15	0.00	3.15	3.97	18.95	19.37	5.09	18.95	19.63	17.21	7.60	18.91	12.39	11.73	17.06
35	3.26	0.00	3.26	4.01	19.15	19.57	5.18	19.15	19.84	17.42	7.74	19.15	12.60	11.80	17.27
34	3.39	0.00	3.39	4.06	19.35	19.77	5.29	19.35	20.06	17.64	7.89	19.51	12.82	11.87	17.47
33	3.51	0.00	3.51	4.10	19.56	19.96	5.39	19.56	20.29	18.18	8.05	19.86	13.05	11.95	17.69
32	3.63	0.00	3.63	4.14	19.77	20.20	5.51	19.77	20.52	18.53	8.21	20.27	13.26	12.02	17.91
31	3.76	0.00	3.76	4.19	19.99	20.42	5.63	19.99	20.76	18.90	8.38	20.67	13.53	12.09	18.14
30	3.90	0.00	3.90	4.24	20.21	20.65	5.76	20.21	21.01	19.25	8.56	21.10	13.79	12.16	18.38
29	4.06	0.00	4.06	4.28	20.44	20.88	5.96	20.44	21.27	19.69	8.75	21.55	14.05	12.23	18.63
28	4.22	0.00	4.22	4.33	20.68	21.13	6.05	20.68	21.54	20.12	8.95	22.02	14.33	12.31	18.89
27	4.39	0.00	4.39	4.39	20.92	21.38	6.21	20.92	21.82	20.57	9.17	22.52	14.62	12.38	19.16
26	4.58	0.00	4.58	4.44	21.17	21.63	6.38	21.17	22.11	21.04	9.39	23.04	14.93	12.45	19.44
25	4.79	0.00	4.79	4.49	21.44	21.90	6.57	21.44	22.42	21.55	9.63	23.61	15.26	12.52	19.74
24	5.01	0.00	5.01	4.55	21.71	22.18	6.77	21.71	22.74	22.09	9.89	24.21	15.61	12.59	20.05
23	5.26	0.00	5.26	4.61	21.99	22.47	6.99	21.99	23.08	22.67	10.17	24.85	15.97	12.66	20.38
22	5.53	0.00	5.53	4.67	22.29	22.77	7.24	22.29	23.43	23.29	10.47	25.53	16.36	12.73	20.73
21	5.82	0.00	5.82	4.74	22.59	23.09	7.50	22.59	23.81	23.95	10.79	26.27	16.77	12.79	21.09
20	6.15	0.00	6.15	4.80	22.92	23.42	7.80	22.92	24.21	24.67	11.14	27.07	17.21	12.86	21.48
19	6.51	0.00	6.51	4.87	23.26	23.76	8.14	23.26	24.64	25.45	11.52	27.93	17.68	12.92	21.90
18	6.92	0.00	6.92	4.95	23.61	24.13	8.51	23.61	25.10	26.31	11.94	28.89	18.20	12.99	22.36
17	7.36	0.00	7.36	5.03	23.99	24.51	8.93	23.99	25.60	27.25	12.41	29.95	18.77	13.05	22.86
16	7.90	0.00	7.90	5.11	24.39	24.92	9.41	24.39	26.15	28.31	12.93	31.12	19.39	13.12	23.41
15	8.50	0.00	8.50	5.20	24.82	25.36	9.9[illegible]	24.82	26.75	29.47	13.52	32.42	20.06	13.19	24.01
14	9.20	0.00	9.20	5.30	25.28	25.83	10.62	25.28	27.42	30.77	14.20	33.89	20.80	13.26	24.66
13	10.01	0.00	10.01	5.40	25.78	26.34	11.38	25.78	28.17	32.25	14.96	35.55	21.61	13.34	25.40
12	10.96	0.00	10.95	5.52	26.32	26.89	12.29	26.32	29.04	33.95	15.90	37.51	22.55	13.45	26.26
11	12.14	0.00	12.14	5.64	26.91	27.50	13.59	26.91	30.06	36.05	16.39	39.86	23.67	13.59	27.29
10	13.57	0.00	13.57	5.78	27.57	28.17	14.75	27.57	31.27	38.57	18.03	42.70	25.01	13.81	28.57
9	15.37	0.00	15.37	5.94	28.32	28.94	16.40	28.32	32.77	41.03	20.02	46.20	26.61	14.17	30.14
8	17.71	0.00	17.71	6.12	29.18	29.82	18.43	29.18	34.00	45.41	22.21	50.55	28.47	14.81	32.09
7	20.85	0.00	20.85	6.33	30.20	30.86	21.79	30.20	37.24	50.23	25.23	56.21	30.66	15.98	34.58
6	25.30	0.00	25.30	6.59	31.45	32.14	26.15	31.45	40.99	56.93	29.02	64.18	33.46	19.19	38.38
5	32.10	0.00	32.10	6.93	33.07	33.79	32.94	33.07	46.60	67.55	36.42	76.74	37.80	22.32	43.30
4	43.09	0.00	43.09	7.41	35.36	36.13	44.32	35.36	56.70	86.83	47.75	99.10	46.21	30.14	55.17
3	67.76	0.00	67.75	9.22	59.21	60.06	66.27	59.21	78.73	127.54	69.25	144.36	65.37	46.07	79.97
2	145.27	0.00	145.27	10.31	49.20	50.27	145.04	49.20	153.72	244.06	126.30	275.34	122.31	89.90	151.80

RMS RADIAL IMPACT ERROR IN METERS DUE TO NEGLECT OF ANOMALOUS GRAVITY IN GUIDANCE AND NAVIGATION

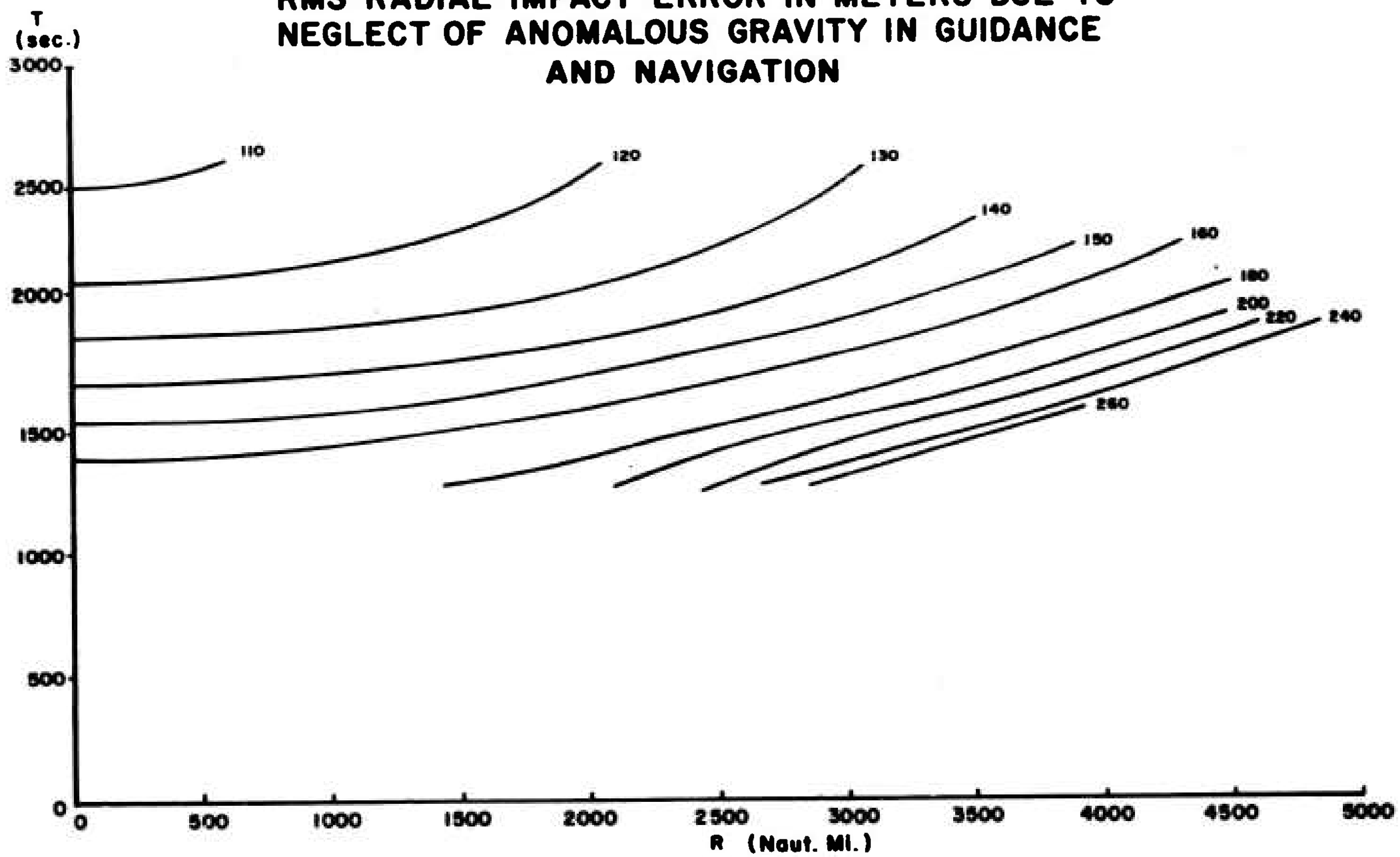

FIGURE 6

RMS RADIAL IMPACT ERROR IN METERS DUE TO NEGLECT OF ANOMALOUS GRAVITY IN GUIDANCE

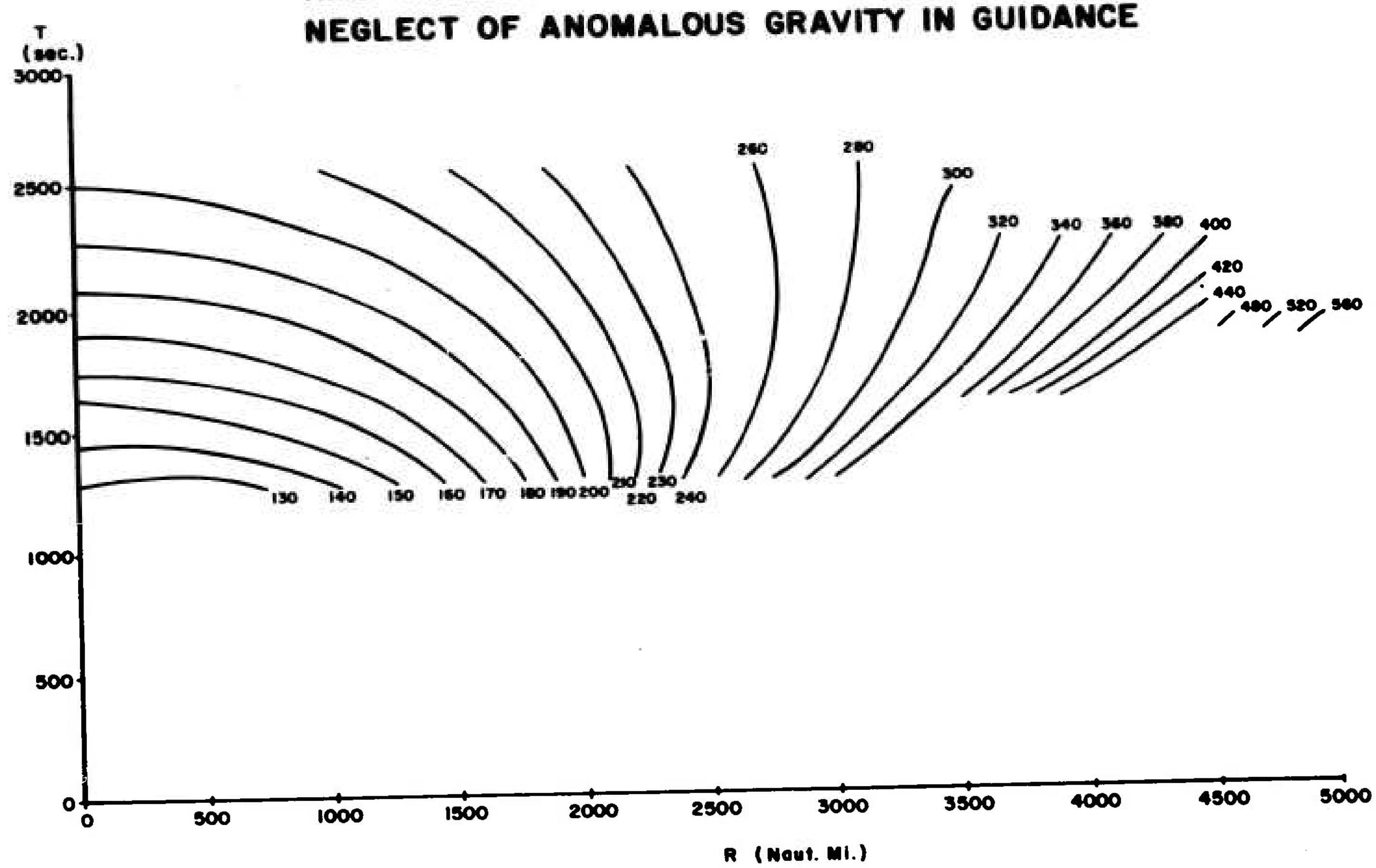

FIGURE 7

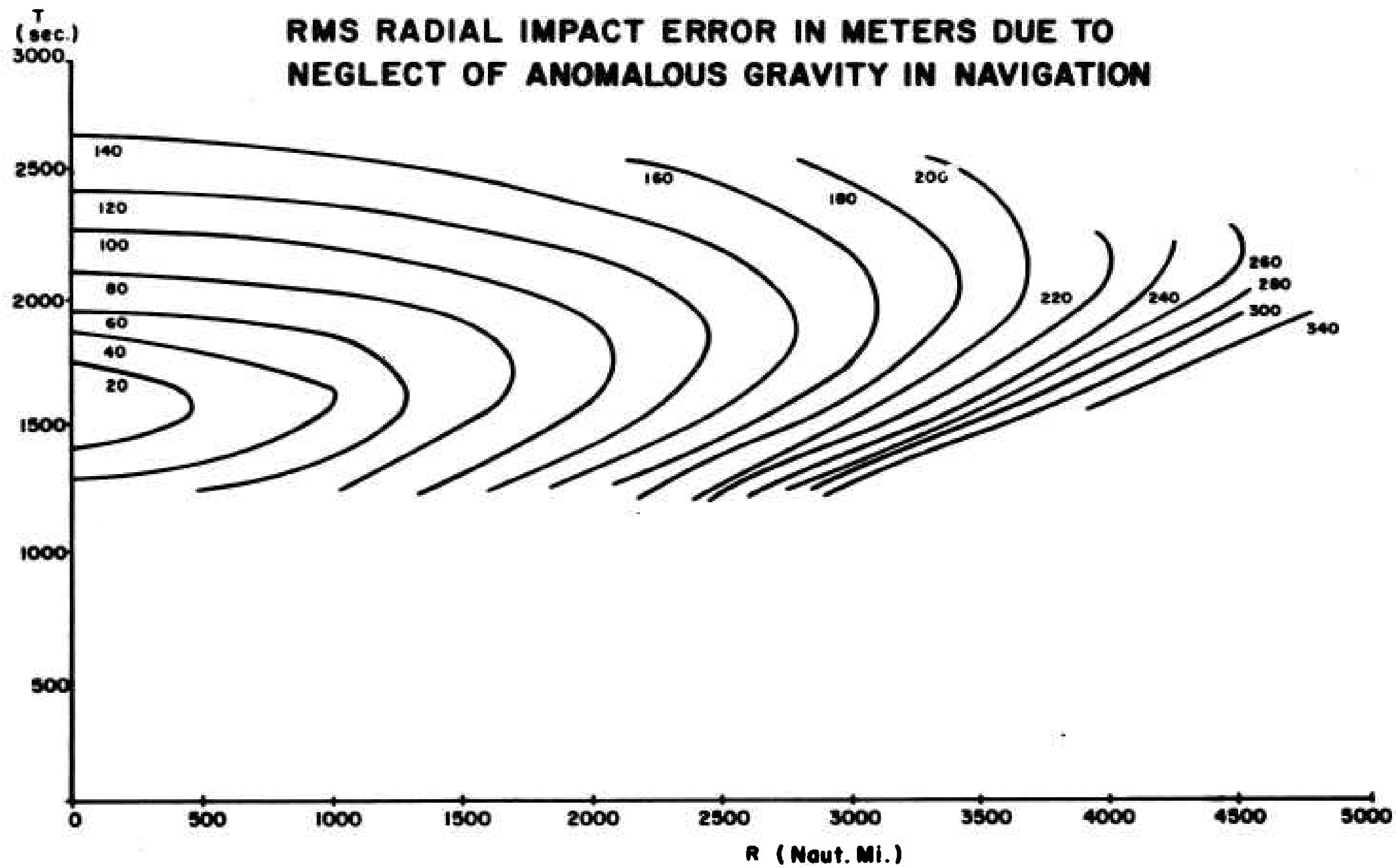
RMS RADIAL IMPACT ERROR IN METERS DUE TO
NEGLECT OF ANOMALOUS GRAVITY IN NAVIGATION
T
(sec.)
3000
2500
2000
1500
1000
500
0
500
1000
1500
2000
2500
3000
3500
4000
4500
5000
R (Naut. Mi.)
20
40
60
80
100
120
140
160
180
200
220
240
260
280
300
340

FIGURE 8

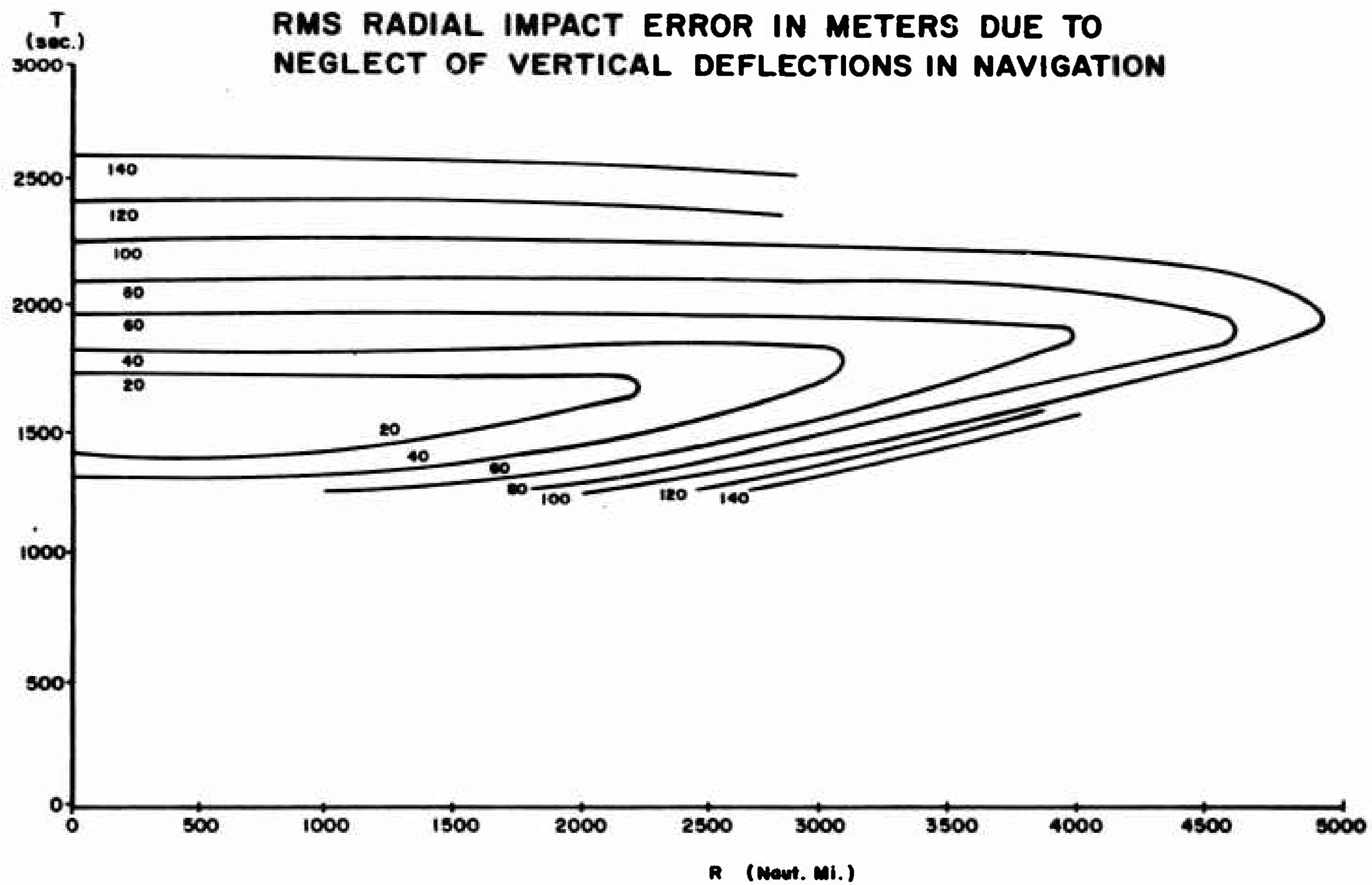
RMS RADIAL IMPACT ERROR IN METERS DUE TO
NEGLECT OF VERTICAL DEFLECTIONS IN NAVIGATION
T
(sec.)
3000
2500
2000
1500
1000
500
0
140
120
100
80
60
40
20
20
40
60
80
100
120
140
0
500
1000
1500
2000
2500
3000
3500
4000
4500
5000
R (Naut. Mi.)

FIGURE 9

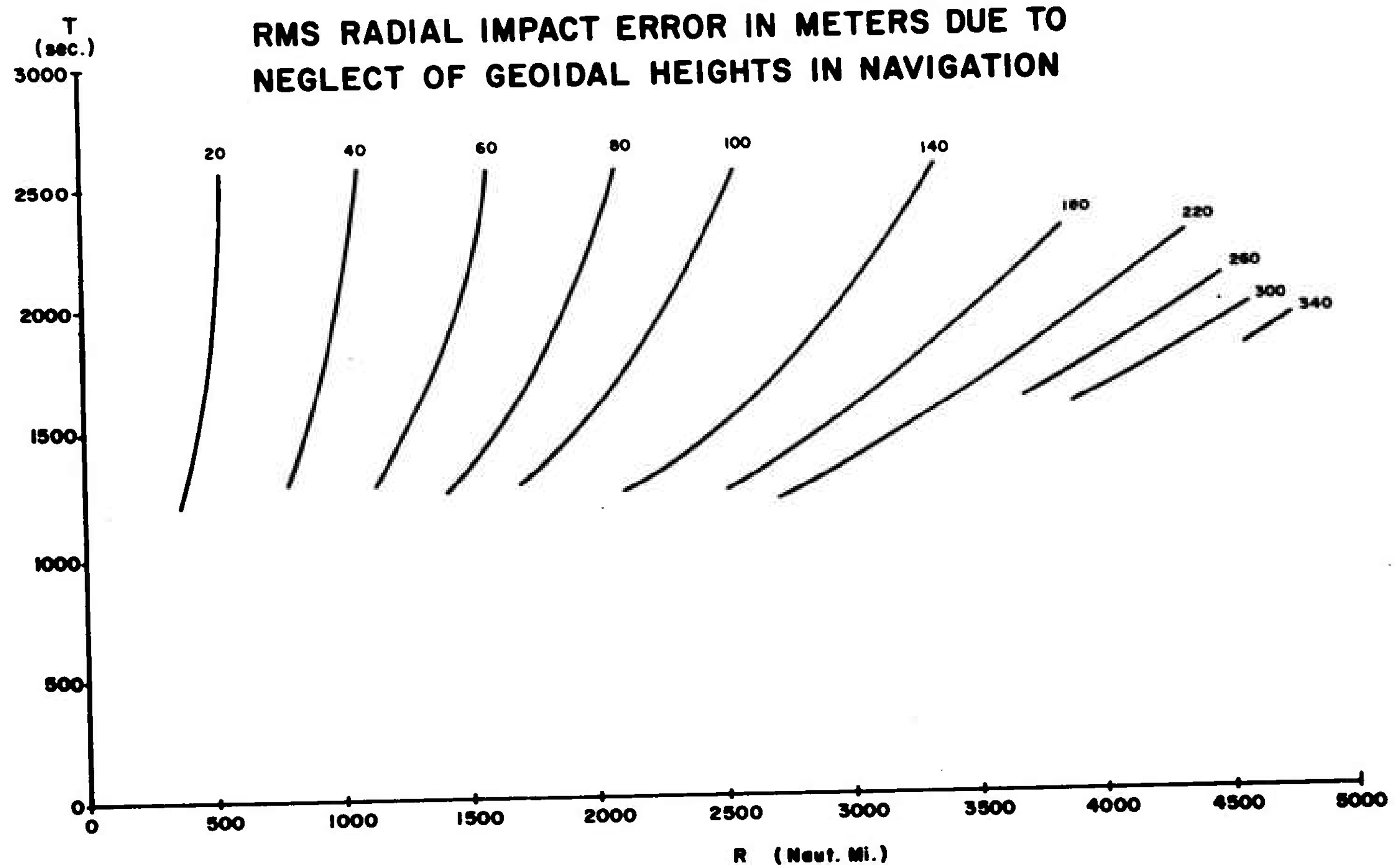
RMS RADIAL IMPACT ERROR IN METERS DUE TO
NEGLECT OF GEOIDAL HEIGHTS IN NAVIGATION
T
(sec.)
3000
2500
2000
1500
1000
500
0
20
40
60
80
100
140
180
220
260
300
340
0
500
1000
1500
2000
2500
3000
3500
4000
4500
5000
R (Naut. Mi.)

FIGURE 10

For the preceding rows of the listing the actual field is again assumed to be that represented by the k_n for $2 \leqslant n \leqslant 174$ and the impact errors result from the omission in the gravity computations of terms of degree n greater than or equal to that indicated in the first column; this omission occurring everywhere, in guidance only, etc.

The radial impact errors for all 45 trajectories resulting from the omission of all terms $n \geqslant 2$ are represented in the error contour charts of Figures 6-10. For given values of R and time of flight T the total radial impact error (in meters) and the contributions to it can be read from these charts.

An interesting conclusion concerning navigation errors can be drawn from Figures 9 and 10. The sketch of Figure 11 represents the superimposed 40 meter error contours from these two charts. For points to the right of the more vertical contour, the neglect of geoidal heights by navigation produces radial impact errors greater than 40 meters. For points to the left of the looped contour, the neglect of vertical deflections by navigation produces radial impact errors less than 40 meters. Hence, in the shaded area the neglect of geoidal heights makes a greater contribution to the impact error than does the neglect of vertical deflections. By plotting the intersections of corresponding contours in Figs. 9 and 10 and joining the points with smooth curves, as in Figure 12, a region is delineated within which the neglect of geoidal heights makes a greater contribution to the impact error than does the neglect of vertical deflections. This region covers a large part of the area representing possible combinations of range and time of flight.

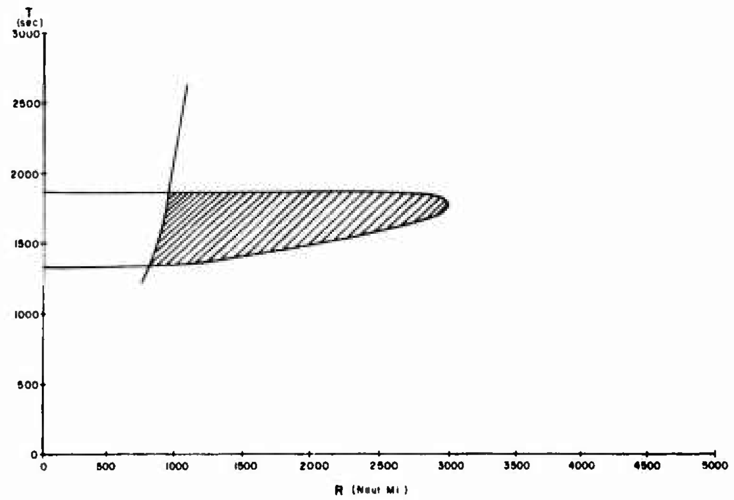

FIGURE 11

At present the navigation system is compensated for vertical deflections in areas where the data are available, but transmits to fire control no output of height above the reference ellipsoid, for which data on geoidal heights would be necessary. Fire control makes some allowance for geoidal height in the computation of target offsets due to tesseral gravity, but makes no allowance for variations in launch depth.

The height of the submarine above the ellipsoid (or depth below it if height < 0) is just as truly a navigation quantity as are the coordinates of latitude and longitude and should properly be accounted for by fire control and missile guidance as one of the three components of initial position, instead of being only partially accounted for in the target offsets.

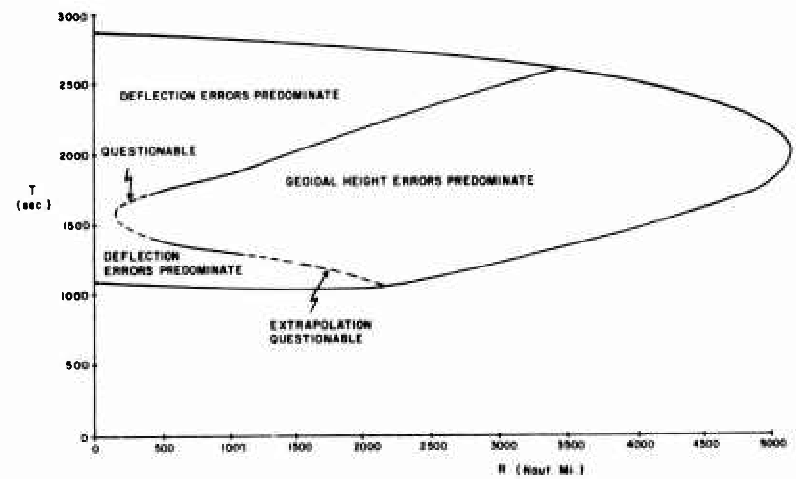

FIGURE 12

It would appear that a comparatively minor modification of the navigation system would allow it to produce and transmit to fire control the height of the submarine above the reference ellipsoid. Figure 13 illustrates the computations involved in a three-channel inertial navigator which employs data from the electromagnetic log to control errors in the horizontal channels and data from the depth gauge to control errors in the vertical channel, the latter in such a manner as to produce an output of height above the reference ellipsoid. The method shown of introducing log data is identical with that now employed in the MK 2 MOD 6 SINS. Also, the method now in use to introduce vertical deflection data into the

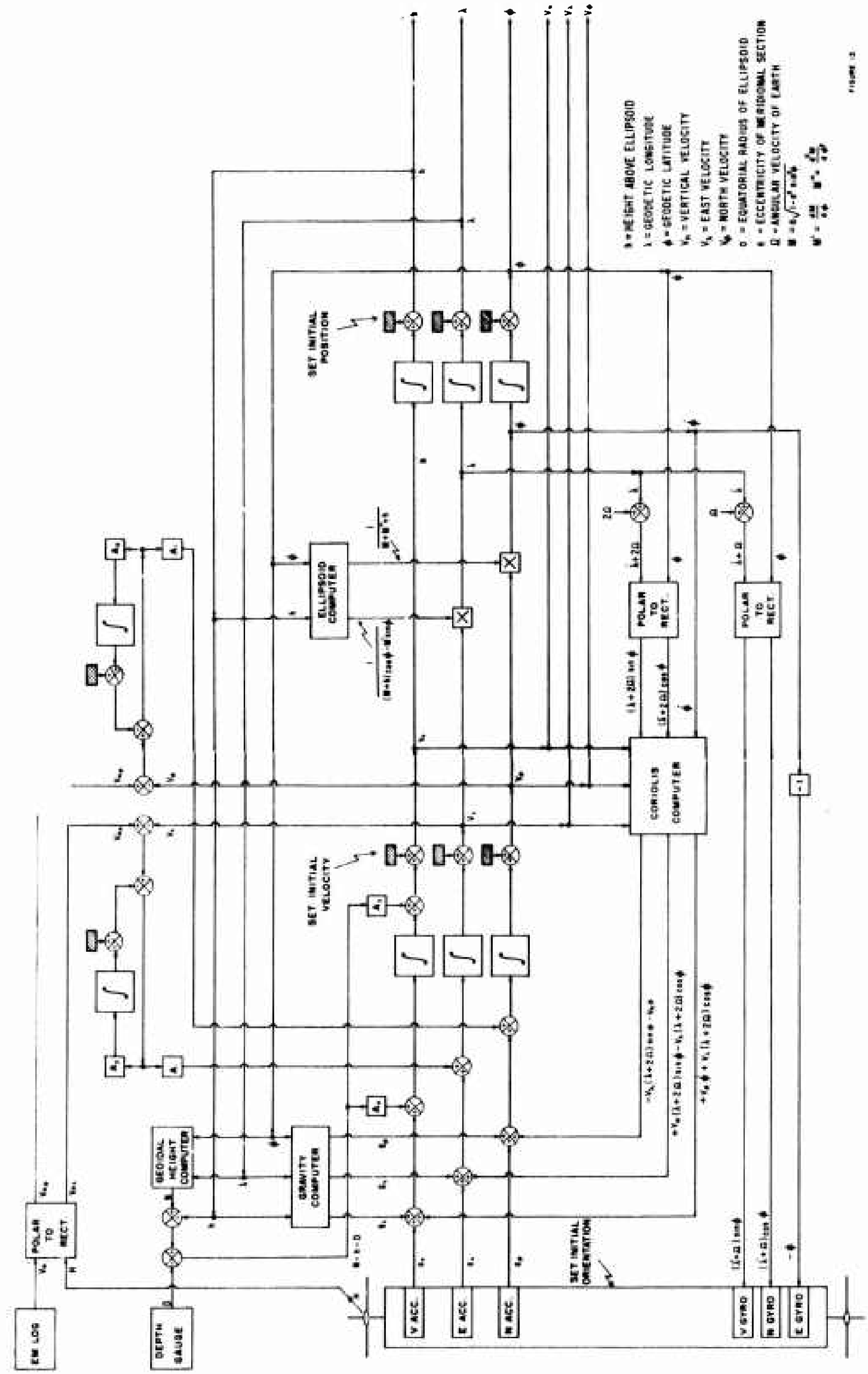
EM LOG
DEPTH GAUGE
POLAR TO RECT.
GEOIDAL HEIGHT COMPUTER
GRAVITY COMPUTER
ELLIPSOID COMPUTER
CORIOLIS COMPUTER
SET INITIAL VELOCITY
SET INITIAL POSITION
SET INITIAL ORIENTATION
V ACC.
E ACC.
N ACC.
V GYRO
N GYRO
E GYRO
h = HEIGHT ABOVE ELLIPSOID
λ = GEODETIC LONGITUDE
φ = GEODETIC LATITUDE
V_V = VERTICAL VELOCITY
V_λ = EAST VELOCITY
V_φ = NORTH VELOCITY
a = EQUATORIAL RADIUS OF ELLIPSOID
e = ECCENTRICITY OF MERIDIONAL SECTION
Ω = ANGULAR VELOCITY OF EARTH

FIGURE 13

SINS is practically equivalent to that indicated in Figure 13, the stored and interpolated values of the deflection components being equivalent to the g_λ and g_φ outputs of the Gravity Computer. The MK 2 MOD 6 SINS also includes an incompletely instrumented vertical channel which is stabilized by the depth gauge and produces an output of vertical velocity $\dot{v}_h$ which is transmitted to fire control, but no output of height h.

This incomplete vertical channel can be regarded as an approximation to that shown in Figure 13. In the approximate instrumentation the Geoidal Height Computer is omitted (equivalent to taking N = 0) and the output of the second integrator in the vertical channel represents approximately the height of the submarine above the geoid, but is not transmitted to fire control. By including the Geoidal Height Computer in the manner indicated in Figure 13, the output of the second integrator would represent height above the ellipsoid and could be transmitted to fire control. Use of this value in the initial position computations would account in the theoretically correct manner both for geoidal heights and for variations in launch depth.

The Geoidal Height Computer would in practice consist merely of a means of storing and interpolating geoidal heights at a set of grid points, exactly similar to the method now used for the vertical deflection components, and could be implemented by storing the geoidal heights on the same tapes as the vertical deflections and subjecting them to the same interpolation process. Furthermore, the basic data required to produce the table of geoidal heights is identical with that required to produce the tables of vertical deflection components, and the computations are very nearly the same.

XII. SPECIALIZATION TO A PARTICULAR LAUNCH AREA

In introducing the statistical treatment adopted in this report it was admitted that a more useful statistical description of the impact errors would be one for which a single target is considered and the launch points are uniformly distributed over some limited launch area. It seems possible to utilize the statistical treatment which has actually been employed to obtain results which are in some respects representative of what would be expected in this more realistic situation.

This extension rests, first, on the observation that the initial position errors depend on the geoidal height and the vertical deflection components at the launch point and, second, on the fact that the in-flight errors depend principally on the geoidal heights over an area which does not extend very far from the launch point. Three reinforcing factors contribute to the truth of the latter statement. First, the effects of the short-wavelength variations in geoidal height which might distinguish one launch area from another are attenuated more rapidly with height than are the effects of the long-wavelength variations and, hence, would be most significant near the launch point and near the target. Second, disturbances of the gravity field near the launch point make a greater contribution to the impact error than do similar disturbances near the target because the missile is moving more slowly near the launch point and is affected by the disturbances for a longer time. Finally, a velocity error produced by a disturbance early in flight has a longer time to act than does a similar velocity error produced late in the flight.

For these reasons it is plausible to assume that the impact errors would not be much different if the disturbances of the gravity field were limited to an area near the launch point than they would be if similar disturbances extended over the entire earth. Hence, the statistical treatment which has been employed in this report can reasonably be interpreted as applying to a specific launch area by employing a set of degree variances k_n which would characterize the geoidal heights over the entire earth if the entire earth were as "disturbed" as is the launch area. There are several objections to this procedure, some of which are unavoidable. First, the short-wavelength variations in geoidal height would produce navigation ve.ocity errors which might be more important than the navigation position errors. Although navigation velocity errors have not been considered here, it seems possible that they could be included and this objection avoided. A second and unavoidable objection to this method lies in the fact that the statistical treatment employed in this report necessarily leads to the conclusion that the mean impact error is zero, and this is almost certainly not the case for missiles launched toward a single target from points of a limited launch area. Finally, and also unavoidably, is the difficulty of quantifying the idea of "near the launch point" and obtaining a set of degree variances which would adequately describe the geoidal heights in such an indefinite area.

It seems worthwhile to digress here to emphasize some fundamental differences between the statistical methods used in this report and the more elaborate techniques which have been used in some previous studies of missile impact errors. These other studies have extended to the treatment of in-flight errors some methods which have been widely employed in navigation error analysis. In these methods, the incomplete or approximate knowledge of the geoidal height function (or other gravity-dependent quantities: gravity anomalies, deflections, etc.) has been represented by modeling the geoidal heights (e.g.), as a stochastic process on the sphere or, more often, in the plane. This view is expressed in Heiskanen and Moritz's ***Physical Geodesy*** by the statement, "It should be mentioned that the mathematics behind the statistical description of the gravity anomalies is the theory of stochastic processes. The gravity anomaly field is treated as a stationary stochastic process on a sphere ··· ". More recently Moritz (***The Role of Statistical Techniques in the Determination of the Earth's Gravitational Field***, International Symposium, 26-30 November, 1973) appears to have modified this point of view by saying "Therefore Moritz proposed to use, instead of an interpretation as a stochastic process, an interpretation in terms of a covariance analysis of individual functions ··· . This also takes into account that in reality there is only one individual earth's gravity field and not a phase space of many such fields."

The elementary statistical methods of this report seem to be more consistent with Moritz's recent views than do those of many previous studies. In particular, the degree variances are regarded here not as statistical quantities (ensemble averages) but merely as a partial description of a completely deterministic (although possibly unknown) geoidal height function. The only properly statistical concepts employed here arise from the consideration of an ensemble of launch points and targets.

Thus, it seems impossible to obtain a completely satisfactory statistical description of the impact errors occurring when missiles are launched toward a specific target from a limited launch area, except in the case when the gravity field is completely known and the statistical description is unnecessary (cf. Heiskanen and Moritz, in a somewhat different connection, "This we obviously do not know; and if we knew it, then the covariance function would have lost most of its significance, because then we could solve our problems rigorously without needing statistics.") With the understanding that it is practically impossible to solve in any precise manner the statistical problem posed by a limited launch area, it nevertheless seems possible to obtain plausible results by adopting a set of degree variances which would constitute a partial description of the geoidal heights of the entire earth if the entire earth were "as rough", in some intuitive and imprecise sense, as is the launch area under consideration. It is planned to make such computations by modifying the values of the parameters involved in Equation (43) after some of the topics discussed in the next section have been investigated.

XIII. LIMITATIONS AND POSSIBLE EXTENSIONS

The preceding analysis and the numerical results have been based on the assumption that the ship remains stationary at the launch point, with the result that the navigation velocity errors are zero. Retaining the maximum value of $n = 174$ that has been used, it would probably not be difficult to extend the analysis and results to include velocity errors for ship's speed below about 10 knots. This follows from the fact that for $n \leqslant 174$ the shortest wavelength of geoidal height which is taken into account is about 120 nautical miles. For a ship's speed of 10 knots, this distance would be traversed in 12 hours and the vertical deflections affecting the navigation system would have this period, or a frequency of about 0.5 rad/hr. This is so far below the Schuler frequency of about 4.5 rad/hr that the vertical indication established by the navigation system is essentially coincident with the plumb line. Hence, the navigation position errors could be computed as before.

The navigation velocity errors in this case would be simply related to the changing direction of the plumb line relative to the geodetic vertical, as influenced by the speed of the ship, and might be accounted for rather easily. The only complication which would be expected to be encountered in this process would become evident in the situation first considered: the nominal launch position being the north pole of the reference sphere and the target in the Meridian of Greenwich. To discuss this complication it is assumed that some constant ship's speed, say 10 knots, will be adopted throughout the investigation and that, if necessary, all of the results will be recomputed for some different ship's speed. Thus, the speed of the ship is not regarded as a parameter in what follows. An additional parameter does, however, enter the problem, for as the ship crosses the north pole it may be moving south along any meridian; that is, the velocity of the ship may form an arbitrary angle δ with the direction to the target, and the navigation velocity error would depend on this angle.

Once the initial velocity error has been determined, there is no difficulty in computing the resulting impact error, since this would depend on the elements of the last three rows of the matrix $X(T, t_0)$. These elements were actually computed in deriving the results which have already been presented, but were not used to obtain those results.

When an arbitrary launch position and target azimuth are considered, it becomes necessary also to consider an arbitrary value of δ. The complex impact error $\rho(\alpha,\beta,\gamma)$ of Equation (36) would then become $\rho(\alpha,\beta,\gamma,\delta)$. To generalize the earlier statistical treatment it would now be necessary to form ensemble averages

over all values of the four parameters $\alpha,\beta,\gamma,\delta$; instead of over values of only the three parameters involved previously. It has not yet been determined whether it will be possible to carry out these operations.

Assuming that the extension of the analysis just outlined can actually be made, it would still not be adequate to treat the case where the speed of the ship is much in excess of 10 knots, or even the 10-knot case if values of n much larger than 174 are to be considered. In both of these cases the frequency components of the vertical deflections would lie so close to the Schuler frequency that it would be necessary to account for the transient response of the navigation system to vertical deflections. It is expected that the height and vertical velocity errors would not be significantly affected by the speed of the ship as a result of the strong control over the errors of the vertical channel exerted by the depth gauge.

It has already been stated that increasing the maximum value of n beyond 174 would be expected to increase the computing time very substantially. This increase comes about principally in computing a_{4n} and b_{4n} of Equations (42), to a lesser extent in computing a_{3n} and b_{3n}, and not at all in computing a_{2n}, b_{2n}, and a_{1n}. An examination of a_{4n} and b_{4n} as functions of $n \leqslant 174$ for one trajectory shows that they increase almost linearly with n. It therefore, seems possible that these quantities could merely be extrapolated to much larger values of n with checks of the extrapolation being made for only a few of the 45 trajectories by actually computing a_{4n} and b_{4n}. The behavior of a_{3n} and b_{3n} appears to be more complicated but, if an extrapolation cannot be employed for these quantities, it would not be difficult to compute them at length, since the time required is approximately proportional to n rather than n^2 as for a_{4n} and b_{4n}. It might be mentioned that the accuracy of such extrapolations could be rather low because the degree variances k_n of any reasonable geoidal height function decrease so rapidly with increasing n.

APPENDIX A

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This report presents an analysis of the impact errors of a ship-launched ballistic missile produced by an anomalous gravity field. All other error sources are assumed to be absent. The normal earth for which the impact errors would be zero is regarded as spherical and non-rotating. The standard atmosphere is accounted for. Both navigation and in-flight effects are considered, the former under the assumption that the ship remains stationary at the launch point.

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The treatment is largely deterministic, with elementary statistical concepts being introduced to derive a description of the impact errors applicable to the case of an unknown anomalous field.

Some numerical statistical results are presented, in the form of a table for a particular trajectory and as graphs for trajectories having various ranges and times of flight.